

Observables

II : Quantum Observables

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Abstract

In this work we discuss the notion of observable - both quantum and classical - from a new point of view. In classical mechanics, an observable is represented as a function (measurable, continuous or smooth), whereas in (von Neumann's approach to) quantum physics, an observable is represented as a bounded selfadjoint operator on Hilbert space. We will show in the present part II and the forthcoming part III of this work that there is a common structure behind these two different concepts. If \mathcal{R} is a von Neumann algebra, a selfadjoint element $A \in \mathcal{R}$ induces a continuous function $f_A : \mathcal{Q}(\mathcal{P}(\mathcal{R})) \rightarrow \mathbb{R}$ defined on the *Stone spectrum* $\mathcal{Q}(\mathcal{P}(\mathcal{R}))$ ([7]) of the lattice $\mathcal{P}(\mathcal{R})$ of projections in \mathcal{R} . f_A is called the observable function corresponding to A . The aim of this part is to study observable functions and its various characterizations.

Für Karin

Contents

1	Introduction	4
2	Quantum Observables	6
2.1	Motivation: The Presheaf of Spectral Families	7
2.2	Basic Properties	15
2.3	Abstract Characterization of Observable Functions	23
2.4	The Gelfand Transformation	42

Chapter 1

Introduction

*“Neue Blicke durch die alten Löcher”
(Georg Christoph Lichtenberg, Aphorismen)*

In von Neumann’s axiomatic approach to quantum physics, an observable property of a quantum system is represented by a selfadjoint bounded linear operator on a suitable complex Hilbert space. In classical mechanics, however, an observable property is, depending on the context, represented by a measurable, continuous or smooth function on phase space. In this and in the next parts ([8, 9]) we will show how to overcome this apparently fundamental difference.

In the previous part ([7]), we have studied the Stone spectrum $\mathcal{Q}(\mathbb{L})$ of a lattice \mathbb{L} . The elements of $\mathcal{Q}(\mathbb{L})$ are the maximal dual ideals in \mathbb{L} . $\mathcal{Q}(\mathbb{L})$ is equipped with a topology by the requirement that the sets

$$\mathcal{Q}_a(\mathbb{L}) := \{\mathfrak{B} \in \mathcal{Q}(\mathbb{L}) \mid a \in \mathfrak{B}\} \quad (a \in \mathbb{L})$$

form a basis of this topology. Of course, this is a manifest generalization of Stone’s construction ([2]). If \mathcal{S} is a presheaf on a complete lattice \mathbb{L} , the sheafification of \mathbb{L} leads to the same construction of $\mathcal{Q}(\mathbb{L})$ as base space of the étale space of \mathcal{S} . Moreover, as we have proved in [7], if \mathbb{L} is the lattice $\mathcal{P}(\mathcal{R})$ of projections in an abelian von Neumann algebra \mathcal{R} , then $\mathcal{Q}(\mathcal{R})$ is homeomorphic to the Gelfand spectrum of \mathcal{R} . In the same way we define a topology on the set $\mathcal{D}(\mathbb{L})$ of all dual ideals in \mathbb{L} . The Stone spectrum $\mathcal{Q}(\mathbb{L})$ is a dense subset of $\mathcal{D}(\mathbb{L})$, but note that, except for trivial situations, the topology of $\mathcal{D}(\mathbb{L})$ is not Hausdorff.

If $E = (E_\lambda)_{\lambda \in \mathbb{R}}$ is a bounded spectral family in a complete lattice \mathbb{L} , we call the function $f_E : \mathcal{D}(\mathbb{L}) \rightarrow \mathbb{R}$, defined by

$$f_E(\mathcal{J}) := \inf\{\lambda \in \mathbb{R} \mid E_\lambda \in \mathcal{J}\},$$

the *observable function corresponding to E* . The aim of this part is to study observable functions and its various characterizations, in particular in the case $\mathbb{L} = \mathcal{P}(\mathcal{R})$ for a von Neumann algebra \mathcal{R} . In that case, bounded spectral families correspond to selfadjoint elements of \mathcal{R} . Hence we write f_A for the observable function corresponding to the spectral family E^A of \mathcal{R}_{sa} . We prove that $f_A : \mathcal{Q}(\mathcal{R}) \rightarrow \mathbb{R}$ is a *continuous* function whose range is the spectrum of A . Moreover, we show that, if \mathcal{R} is abelian, the mapping $A \mapsto f_A$ coincides with the Gelfand transformation. Thus it is tempting to regard $\mathcal{Q}(\mathcal{R})$ as a sort of phase space. This interesting question will be discussed in part IV ([9]).

We refer to the introduction of part I ([7]) for a detailed description of results of the present part.

Chapter 2

Quantum Observables

Observables in *quantum physics* are selfadjoint operators of an appropriate Hilbert space \mathcal{H} . Physically meaningful is not the *precise value* of an observable (which is an inconsistent notion in quantum theory by the Kochen - Specker theorem ([12])) but its *expectation value* in a given state of the physical system. In quantum theory the expectation that the value of the observable A lies in the Borel set $\Delta \subseteq \mathbb{R}$ when the physical system is in the pure state $x \in \mathcal{H}$ is given by

$$\langle E(\Delta)x, x \rangle = \int_{\Delta} \lambda d \langle E_{\lambda}x, x \rangle$$

where $E = (E_{\lambda})_{\lambda \in \mathbb{R}}$ is the spectral resolution of the selfadjoint operator A . So the essence of an observable is its spectral family. Later on we will show how to describe also classical observables by spectral families. Of course one can object that one can easily perform algebraic operations on operators and functions but it is an intricate problem to describe these operations in the language of spectral families. From the physical point of view however, the possibility of adding two given observables to obtain a new one is merely a mathematical reflex: what is the meaning of the sum of the position and the momentum operator or the sum of two different spin operators?

The aim of the following section is to show how the representation of quantum observables as observable functions evolves from sheaf theoretical considerations.

2.1 Motivation: The Presheaf of Spectral Families

We know that there is only the trivial sheaf on the quantum lattice $\mathbb{L}(\mathcal{H})$ of closed subspaces of the Hilbert space \mathcal{H} . But what about presheaves?

An obvious example is the following one: For $U \in \mathbb{L}(\mathcal{H})$ let $\mathcal{S}(U) := \mathcal{L}(U)$ be the space of bounded linear operators $U \rightarrow U$ and for $V \in \mathbb{L}(\mathcal{H})$, $V \subseteq U$, we define a “restriction map”

$$\rho_V^U : \mathcal{L}(U) \rightarrow \mathcal{L}(V)$$

by

$$\rho_V^U(A) := P_V A |_V .$$

Clearly these data give a presheaf on $\mathbb{L}(\mathcal{H})$.

This example looks somewhat artificial because the restriction maps defined above do not coincide with the usual idea of restricting a mapping from its domain to a smaller set. But we will see in part III, that it leads quite naturally to the notion of *Positive Operator Valued Measures*. The elements of the stalks of this presheaf, however, have a quantum mechanical interpretation.

Remark 2.1 *Let $A \in \mathcal{L}(U)$ and let $\mathfrak{B}_{\mathbb{C}x} \in \mathcal{Q}_U(\mathcal{H})$ be an atomic quasipoint (in $\mathbb{L}(U)$). Then the germ of A in $\mathfrak{B}_{\mathbb{C}x}$ is given by $\langle Ax, x \rangle$, where $x \in S^1(\mathcal{H}) \cap \mathbb{C}x$.*

Namely, if $A, B \in \mathcal{L}(U)$, then $A \sim_{\mathfrak{B}_{\mathbb{C}x}} B$ if and only if $P_{\mathbb{C}x} A P_{\mathbb{C}x} = P_{\mathbb{C}x} B P_{\mathbb{C}x}$. Now if $x \in S^1(\mathcal{H})$ then

$$\forall z \in \mathcal{H} : P_{\mathbb{C}x} A P_{\mathbb{C}x} z = \langle Ax, x \rangle \langle z, x \rangle x .$$

Hence $P_{\mathbb{C}x} A P_{\mathbb{C}x} = P_{\mathbb{C}x} B P_{\mathbb{C}x}$ if and only if $\langle Ax, x \rangle = \langle Bx, x \rangle$.

If A is a selfadjoint operator and $x \in S^1(\mathcal{H})$ is in the domain of A , then $\langle Ax, x \rangle$ is interpreted as the *expectation value of the observable A when the quantum mechanical system is in the pure state $\mathbb{C}x$* .

In order to obtain an example of a presheaf on $\mathbb{L}(\mathcal{H})$ whose restriction maps are defined analogously to the ordinary restriction of functions, we shall reformulate the operation of restricting a continuous function $f : U \rightarrow \mathbb{R}$ to an open subset $V \subseteq U$ in the language of lattice theory.

Let M and N be regular Hausdorff spaces. A continuous mapping $f : M \rightarrow N$ induces a lattice homomorphism

$$\begin{aligned}\Phi_f : \mathcal{T}(N) &\rightarrow \mathcal{T}(M) \\ W &\mapsto \bar{f}^{-1}(W)\end{aligned}$$

that is left continuous:

$$\Phi_f\left(\bigcup_{i \in I} W_i\right) = \bigcup_{i \in I} \Phi_f(W_i)$$

for each family $(W_i)_{i \in I}$ in $\mathcal{T}(N)$. Conversely:

Theorem 2.1 *Each left continuous lattice homomorphism $\Phi : \mathcal{T}(N) \rightarrow \mathcal{T}(M)$ induces a unique continuous mapping $f : M \rightarrow N$ such that $\Phi = \Phi_f$.*

The proof is based on the observation that for any point \mathfrak{p} in $\mathcal{T}(M)$ the inverse image $\bar{\Phi}^{-1}(\mathfrak{p})$ is a point in $\mathcal{T}(N)$. Because the points in $\mathcal{T}(M)$ correspond to the elements of M , this gives a mapping $f : M \rightarrow N$. It is then easy to show that f has the required properties.

Now we can describe the restriction of a continuous mapping $f : M \rightarrow N$ to an open subset U of M in the following way:

Proposition 2.1 *Let $f : M \rightarrow N$ be a continuous mapping between regular Hausdorff spaces, $\Phi_f : \mathcal{T}(N) \rightarrow \mathcal{T}(M)$ the left-continuous lattice homomorphism induced by f , and U an open subset of M . Then*

$$\begin{aligned}\Phi_f^U : \mathcal{T}(N) &\rightarrow \mathcal{T}(U) \\ W &\mapsto \Phi_f(W) \cap U\end{aligned}$$

is a left-continuous lattice homomorphism and the corresponding continuous mapping $U \rightarrow N$ is the restriction of f to U .

Proof: $\Phi := \Phi_f^U : \mathcal{T}(N) \rightarrow \mathcal{T}(U)$ is a left-continuous lattice homomorphism, since $\mathcal{T}(M)$ is a completely distributive lattice. Let $\mathfrak{p}_x \subseteq \mathcal{T}(U)$ be the point corresponding to $x \in U$. Then

$$\bar{\Phi}^{-1}(\mathfrak{p}_x) = \{V \in \mathcal{T}(N) \mid x \in \Phi_f(V) \cap U\} = \{V \in \mathcal{T}(N) \mid x \in \Phi_f(V)\} = \mathfrak{p}_{f(x)},$$

where $\mathfrak{p}_{f(x)}$ denotes the point in $\mathcal{T}(N)$ that corresponds to $f(x)$. Hence $\Phi = \Phi_{f|_U}$. \square

Let \mathcal{H} be a Hilbert space. The observables of a quantum mechanical system are selfadjoint operators of \mathcal{H} . Equivalently we can think of observables as spectral families in the lattice $\mathcal{P}(\mathcal{L}(\mathcal{H}))$ of all orthogonal projections in $\mathcal{L}(\mathcal{H})$. Let $\mathcal{S}(\mathcal{H})$ be the set of all spectral families in $\mathcal{P}(\mathcal{L}(\mathcal{H}))$. To begin with, we restrict our attention to those spectral families $E : \mathbb{R} \rightarrow \mathcal{P}(\mathcal{L}(\mathcal{H}))$ that are *bounded from above* :

$$\exists \lambda \in \mathbb{R} : E_\lambda = I.$$

We want to show that the set $\mathcal{S}^{ub}(\mathcal{H})$ of all *upper bounded* spectral families in $\mathcal{P}(\mathcal{L}(\mathcal{H}))$ induces canonically a presheaf $\mathcal{S}_{\mathcal{H}}^{ub}$ on $\mathcal{P}(\mathcal{L}(\mathcal{H}))$. We can perform the construction for an arbitrary orthomodular lattice.

Let \mathbb{L} be a complete lattice and for $a \in \mathbb{L}$ let

$$\mathbb{L}_a := \{b \in \mathbb{L} \mid b \leq a\}.$$

\mathbb{L}_a is a complete orthomodular sublattice (in fact a principle ideal) of \mathbb{L} with maximal element a . We denote by $\mathcal{S}^{ub}(a)$ the set of all spectral families $E : \mathbb{R} \rightarrow \mathbb{L}_a$ that are bounded from above:

$$\exists \lambda \in \mathbb{R} : E_\lambda = a.$$

For $a, b \in \mathbb{L}$, $a \leq b$ we define a restriction mapping

$$\begin{array}{ccc} \varrho_a^b : \mathcal{S}^{ub}(b) & \rightarrow & \mathcal{S}^{ub}(a) \\ E & \mapsto & E^a \end{array}$$

by

$$\forall \lambda \in \mathbb{R} : E_\lambda^a := E_\lambda \wedge a.$$

Obviously,

$$\mathcal{S}_{\mathbb{L}}^{ub} := (\mathcal{S}^{ub}(a), \varrho_a^b)_{a \leq b}$$

is a presheaf on \mathbb{L} . We call it the **spectral presheaf on \mathbb{L}** .

Remark 2.2 *If \mathbb{L} is a lattice of finite type ([7]), the condition of upper boundedness is not necessary. If, in particular, \mathbb{L} is the projection lattice $\mathcal{P}(\mathcal{R})$ of a finite von Neumann algebra, the restriction maps are defined for arbitrary spectral families. Theorem 3.1 in [7] and its proof show that the converse is true too.*

We can make the connection of restricting spectral families to the restriction of continuous real valued functions on a Hausdorff space even more transparent: Let $f : M \rightarrow \mathbb{R}$ be a continuous function on a Hausdorff space M . Then

$$\forall \lambda \in \mathbb{R} : E_\lambda := \text{int}(f^{-1}([-\infty, \lambda]))$$

defines a spectral family $E : \mathbb{R} \rightarrow \mathcal{T}(M)$. (The natural guess for defining a spectral family corresponding to f would be

$$\lambda \mapsto f^{-1}([-\infty, \lambda]).$$

In general, this is only a *pre-spectral family*: it satisfies all properties of a spectral family, except continuity from the right. This is cured by *spectralization*, i.e. by the switch to

$$\lambda \mapsto \bigwedge_{\mu > \lambda} f^{-1}([-\infty, \mu]).$$

But

$$\bigwedge_{\mu > \lambda} f^{-1}([-\infty, \mu]) = \text{int}\left(\bigcap_{\mu > \lambda} f^{-1}([-\infty, \mu])\right) = \text{int}(f^{-1}([-\infty, \lambda])),$$

which shows that our original definition is the natural one.)

One can show that

$$\forall x \in M : f(x) = \inf\{\lambda \mid x \in E_\lambda\},$$

so one can recover the function f from its spectral family E . Let $U \in \mathcal{T}(M)$, $U \neq \emptyset$. Then

$$E_\lambda \cap U = \text{int}\{x \in U \mid f(x) \leq \lambda\} = \text{int}(f|_U^{-1}([-\infty, \lambda])),$$

hence E^U is the spectral family of the usual restriction $f|_U : U \rightarrow \mathbb{R}$ of f to U . We shall investigate the interplay between spectral families in $\mathcal{T}(M)$ and continuous functions $f : M \rightarrow \mathbb{R}$ extensively in the next part.

Let \mathcal{H} be a Hilbert space. The following simple example shows that the restriction E^P of a spectral family E in $\mathcal{P}(\mathcal{L}(\mathcal{H}))$ that is not bounded from above may fail to be a spectral family in $\mathcal{P}(P\mathcal{H})$.

Example 2.1 Let \mathcal{H} be a separable Hilbert space and $(e_n)_{n \in \mathbb{N}}$ an orthonormal basis of \mathcal{H} . Then

$$E_\lambda := \bigvee_{n \leq \lambda} P_{\mathbb{C}e_n} \quad (\lambda \in \mathbb{R})$$

defines a spectral family in $\mathbb{L}(\mathcal{H})$. One can show that this spectral family corresponds (up to some scaling) to the Hamilton operator of the harmonic oscillator. Take $x \in S^1(\mathcal{H})$ such that

$$\forall n \in \mathbb{N} : \langle x, e_n \rangle \neq 0.$$

This means that $P_{\mathbb{C}x} \not\leq E_\lambda$ for all $\lambda \in \mathbb{R}$ and hence

$$E_\lambda^{P_{\mathbb{C}x}} = E_\lambda \wedge P_{\mathbb{C}x} = 0$$

for all $\lambda \in \mathbb{R}$. Therefore

$$\bigvee_{\lambda \in \mathbb{R}} E_\lambda^{P_{\mathbb{C}x}} = 0 \neq P_{\mathbb{C}x}.$$

Remark 2.3 Of course we can drop the requirement

$$\bigvee_{\lambda \in \mathbb{R}} E_\lambda = I$$

in the definition of spectral families. Then we obtain the notion of a **generalized spectral family**. Operators that are given by generalized spectral families are not necessarily densely defined, but their domain of definition is only dense in the closed subspace $\bigvee_{\lambda \in \mathbb{R}} E_\lambda \mathcal{H}$ of \mathcal{H} .

Let us consider the restriction of a spectral family $E : \mathbb{R} \rightarrow \mathcal{P}(\mathcal{L}(\mathcal{H}))$ to $\mathcal{P}(P_{\mathbb{C}x} \mathcal{H})$ more closely. If $P_{\mathbb{C}x} \leq E_\lambda$ for some $\lambda \in \mathbb{R}$, then the hermitian operator corresponding to the spectral family

$$E^{P_{\mathbb{C}x}} : \mathbb{R} \rightarrow \mathcal{P}(P_{\mathbb{C}x} \mathcal{H})$$

is a (real) scalar multiple cI_1 of the identity $I_1 : P_{\mathbb{C}x} \mathcal{H} \rightarrow P_{\mathbb{C}x} \mathcal{H}$. Now $\mathcal{P}(P_{\mathbb{C}x} \mathcal{H}) = \{0, P_{\mathbb{C}x}\}$, hence

$$E_\lambda^{P_{\mathbb{C}x}} = \begin{cases} 0 & \text{for } \lambda < c \\ P_{\mathbb{C}x} & \text{for } \lambda \geq c \end{cases}$$

and

$$c = \inf\{\lambda \in \mathbb{R} \mid P_{\mathbb{C}x} \leq E_\lambda\}.$$

Using the convention

$$\inf \emptyset = \infty$$

we obtain in this way a function on the projective Hilbert space $\mathbb{P}\mathcal{H}$ with values in $\mathbb{R} \cup \{\infty\}$,

$$f_E : \mathbb{P}\mathcal{H} \rightarrow \mathbb{R} \cup \{\infty\},$$

defined by

$$f_E(\mathbb{C}x) := \inf\{\lambda \in \mathbb{R} \mid P_{\mathbb{C}x} \leq E_\lambda\}.$$

Clearly, if E is bounded from above then f_E is bounded from above, too. Moreover, f_E is a bounded function if and only if E is a bounded spectral family, i.e. the corresponding selfadjoint operator A_E is bounded.

The canonical topology on projective Hilbert space $\mathbb{P}\mathcal{H}$ is the quotient topology defined by the projection

$$\begin{aligned} pr : \mathcal{H} \setminus \{0\} &\rightarrow \mathbb{P}\mathcal{H} \\ x &\mapsto \mathbb{C}x. \end{aligned}$$

This means that a subset $\mathcal{W} \subseteq \mathbb{P}\mathcal{H}$ is open if and only if $\bar{pr}^{-1}(\mathcal{W})$ is an open subset of $\mathcal{H} \setminus \{0\}$.

The function f_σ has some remarkable properties:

Proposition 2.2 *Let $E : \mathbb{R} \rightarrow \mathcal{P}(\mathcal{L}(\mathcal{H}))$ be a spectral family and let*

$$f_E : \mathbb{P}\mathcal{H} \rightarrow \mathbb{R} \cup \{\infty\}$$

be the function defined by

$$f_E(\mathbb{C}x) := \inf\{\lambda \in \mathbb{R} \mid P_{\mathbb{C}x} \leq E_\lambda\}.$$

Then

- (1) f_E is lower semicontinuous on $\mathbb{P}\mathcal{H}$;
- (2) if $\mathbb{C}x, \mathbb{C}y, \mathbb{C}z$ are elements of $\mathbb{P}\mathcal{H}$ such that $\mathbb{C}z \subseteq \mathbb{C}x + \mathbb{C}y$, then

$$f_E(\mathbb{C}z) \leq \max(f_E(\mathbb{C}x), f_E(\mathbb{C}y));$$

- (3) $\bar{pr}^{-1}(f_E^{-1}(\mathbb{R}))$ is dense in $\mathbb{P}\mathcal{H}$.

Lower semicontinuity follows from

$$\bar{pr}^{-1}(f_E^{-1}([\lambda, \infty])) \cup \{0\} = E_\lambda \mathcal{H};$$

for then $f_E^{-1}([\lambda, \infty])$ is closed in $\mathbb{P}\mathcal{H}$ for all $\lambda \in \mathbb{R}$ and therefore f_E is lower semicontinuous. The two other properties are obvious from the definitions.

Definition 2.1 A function $f : \mathbb{P}\mathcal{H} \rightarrow \mathbb{R} \cup \{\infty\}$ is called an **observable function** if it has the following properties:

- (1) f is lower semicontinuous;
- (2) if $\mathbb{C}x, \mathbb{C}y, \mathbb{C}z$ are elements of $\mathbb{P}\mathcal{H}$ such that $\mathbb{C}z \subseteq \mathbb{C}x + \mathbb{C}y$ then

$$f(\mathbb{C}z) \leq \max(f(\mathbb{C}x), f(\mathbb{C}y));$$

- (3) $f^{-1}(\mathbb{R})$ is dense in $\mathbb{P}\mathcal{H}$.

The point is that we can reconstruct spectral families in $\mathcal{P}(\mathcal{L}(\mathcal{H}))$ from observable functions on $\mathbb{P}\mathcal{H}$:

Theorem 2.2 The mapping $E \mapsto f_E$ is a bijection from the set of spectral families in $\mathcal{P}(\mathcal{L}(\mathcal{H}))$ onto the set of observable functions on $\mathbb{P}\mathcal{H}$. This mapping is compatible with restrictions:

$$f_{E^P} = f_E|_{\mathbb{P}P\mathcal{H}}.$$

Moreover, $E \in \mathcal{S}^{ub}(\mathcal{H})$ if and only if $f_E^{-1}(\mathbb{R}) = \mathbb{P}\mathcal{H}$, and E belongs to $\mathcal{S}^b(\mathcal{H})$, the set of bounded spectral families in $\mathcal{P}(\mathcal{L}(\mathcal{H}))$, if and only if f_E is bounded.

Sketch of proof: The construction of a spectral family from an observable function f is roughly as follows: for $\lambda \in \mathbb{R}$ let

$$E_\lambda := \overline{\text{pr}}^{-1}(f^{-1}([-\infty, \lambda])) \cup \{0\}.$$

Property (1) assures that E_λ is closed in \mathcal{H} and property (2) implies that E_λ is a subspace of \mathcal{H} . It is not difficult to show that $E : \lambda \mapsto E_\lambda$ is a spectral family in $\mathcal{P}(\mathcal{L}(\mathcal{H}))$ and that

$$f_E = f$$

holds. It follows from Baire's category theorem that $E \in \mathcal{S}^{ub}(\mathcal{H})$ if and only if $f_E^{-1}(\mathbb{R}) = \mathbb{P}\mathcal{H}$. \square

If A is the selfadjoint operator corresponding to the spectral family E , then we also write f_A instead of f_E . The **spectrum** $sp(A)$ of a selfadjoint operator on \mathcal{H} is given by the corresponding observable function f_A in a surprisingly simple manner:

Proposition 2.3 *Let A be a selfadjoint operator on \mathcal{H} . Then*

$$sp(A) = \overline{f_A^{-1}(\mathbb{R})},$$

which simplifies to

$$sp(A) = \overline{f_A(\mathbb{P}\mathcal{H})}$$

if A is bounded from above.

In the next section we will obtain a stronger result (theorem 2.3) for bounded selfadjoint operators.

$\mathcal{O}(\mathcal{H})$ be the set of observable functions on $\mathbb{P}\mathcal{H}$, $\mathcal{O}^{ub}(\mathcal{H})$ the set of observable functions that are bounded from above, and $\mathcal{O}^b(\mathcal{H})$ the set of bounded observable functions.

Let $f \in \mathcal{O}(\mathcal{H})$, $\mathbb{C}x \in \overline{f}^{-1}(\mathbb{R})$ and $\mathfrak{B}_{\mathbb{C}x} \in \mathcal{Q}(\mathcal{H})$ the atomic quasipoint defined by $\mathbb{C}x$. Let further E be the spectral family corresponding to f . Then

$$\begin{aligned} f(\mathbb{C}x) &= \inf\{\lambda \in \mathbb{R} \mid P_{\mathbb{C}x} \leq E_\lambda\} \\ &= \inf\{\lambda \in \mathbb{R} \mid E_\lambda \in \mathfrak{B}_{\mathbb{C}x}\}. \end{aligned}$$

Using this formulation, we can extend the definition of observable functions to arbitrary quasipoints in $\mathcal{P}(\mathcal{L}(\mathcal{H}))$:

Definition 2.2 *Let $f \in \mathcal{O}(\mathcal{H})$ and let $E_f : \mathbb{R} \rightarrow \mathcal{P}(\mathcal{L}(\mathcal{H}))$ be the spectral family corresponding to f . The function*

$$\hat{f} : \mathcal{Q}(\mathcal{H}) \rightarrow \mathbb{R} \cup \{-\infty, \infty\},$$

defined by

$$\hat{f}(\mathfrak{B}) := \inf\{\lambda \in \mathbb{R} \mid E_\lambda \in \mathfrak{B}\},$$

*is called the **observable function on $\mathcal{Q}(\mathcal{H})$ induced by f** .*

Note that $\hat{f}(\mathfrak{B}) = -\infty$ if and only if E is not bounded from below and \mathfrak{B} contains $\{E_\lambda \mid \lambda \in \mathbb{R}\}$.

The observable function \hat{f} induced by $f \in \mathcal{O}^b(\mathcal{H})$ can also be expressed directly in terms of f :

Proposition 2.4 *Let f be a bounded observable function. Then the observable function \hat{f} induced by f is given by*

$$\forall \mathfrak{B} \in \mathcal{Q}(\mathcal{H}) : \hat{f}(\mathfrak{B}) = \inf_{P \in \mathfrak{B}} \sup_{P_{\mathbb{C}x} \leq P} f(\mathbb{C}x).$$

From now on we will denote the observable function $\mathcal{Q}(\mathcal{H}) \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ induced by $f \in \mathcal{O}(\mathcal{H})$ also with the letter f .

Next we will show how observable functions $f : \mathcal{Q}(\mathcal{H}) \rightarrow \mathbb{R} \cup \{-\infty\}$ can be used to assign a value to the germ $[E]_{\mathfrak{B}}$ of a spectral family E in the quasipoint $\mathfrak{B} \in \mathcal{Q}(\mathcal{H})$. We recall that spectral families $E \in \mathcal{S}^{ub}(P)$ and $F \in \mathcal{S}^{ub}(Q)$ are equivalent at the quasipoint $\mathfrak{B} \in \mathcal{Q}_{P \wedge Q}(\mathcal{H})$ if and only if there is an element $R \in \mathfrak{B}$ such that $R \leq P \wedge Q$ and $E^R = F^R$ holds.

Proposition 2.5 *Let $E \in \mathcal{S}^{ub}(P)$, $F \in \mathcal{S}^{ub}(Q)$ be spectral families with corresponding observable functions f_E and f_F respectively. If E and F are equivalent at $\mathfrak{B} \in \mathcal{Q}_{P \wedge Q}(\mathcal{H})$, then*

$$f_E(\mathfrak{B}) = f_F(\mathfrak{B})$$

holds.

This follows directly from the observation that the definition of equivalence at \mathfrak{B} implies

$$\{\lambda \in \mathbb{R} \mid E_\lambda \in \mathfrak{B}\} = \{\lambda \in \mathbb{R} \mid F_\lambda \in \mathfrak{B}\}.$$

The proposition shows that we obtain a mapping

$$v : \mathcal{E}(\mathcal{S}_{\mathcal{H}}^{ub}) \rightarrow \mathbb{R} \cup \{-\infty\}$$

defined by

$$v([E]_{\mathfrak{B}}) = f_E(\mathfrak{B})$$

on the etale space $\mathcal{E}(\mathcal{S}_{\mathcal{H}}^{ub})$. $v([E]_{\mathfrak{B}})$ is called the **value of the germ** $[E]_{\mathfrak{B}}$.

2.2 Basic Properties

Let \mathcal{H} be a Hilbert space and let A be a selfadjoint (not necessarily bounded) operator of \mathcal{H} . Our basic definition is

Definition 2.3 *Let $E^A = (E_\lambda^A)_{\lambda \in \mathbb{R}}$ be the spectral family of A . The function*

$$f_A : \mathcal{Q}(\mathcal{H}) \rightarrow \mathbb{R} \cup \{-\infty, \infty\},$$

defined by

$$f_A(\mathfrak{B}) := \inf\{\lambda \in \mathbb{R} \mid E_\lambda^A \in \mathfrak{B}\},$$

*is called the **observable function corresponding to A** .*

Remark 2.4 *The value ∞ occurs for f_A if and only if $E_\lambda^A \neq I$ for all $\lambda \in \mathbb{R}$, i.e. if E^A is not bounded from above. Analogously the value $-\infty$ occurs if and only if E^A is not bounded from below. A is bounded if and only if the observable function f_A is bounded. Note that this is already the case if f_A is real valued.*

In the following let \mathcal{R} be a von Neumann algebra considered, as a subalgebra of $\mathcal{L}(\mathcal{H})$ for some Hilbert space \mathcal{H} , such that the unit element of \mathcal{R} is the identity operator $I = id_{\mathcal{H}} \in \mathcal{L}(\mathcal{H})$. \mathcal{R}_{sa} denotes the set of selfadjoint elements of \mathcal{R} and $\mathcal{P}(\mathcal{R})$ the lattice of projections in \mathcal{R} . Let $\mathcal{Q}(\mathcal{R})$ be the *Stone spectrum* of \mathcal{R} , i.e. the Stone spectrum of the complete lattice $\mathcal{P}(\mathcal{R})$. If $A \in \mathcal{R}_{sa}$, we denote by $sp(A)$ the *spectrum* of A and by E^A the *spectral family* of A ¹.

In the following two sections we will generalize the results of the foregoing section to arbitrary von Neumann algebras.

Theorem 2.3 *Let $A \in \mathcal{R}_{sa}$ and let $f_A : \mathcal{Q}(\mathcal{R}) \rightarrow \mathbb{R}$ be the observable function corresponding to A . Then*

$$imf_A = sp(A).$$

Proof: The spectrum $sp(A)$ of A consists of all $\lambda \in \mathbb{R}$ such that the spectral family E^A of A is non-constant on every neighbourhood of λ . Assume that $\lambda_0 \in imf_A$, but $\lambda_0 \notin sp(A)$. Then there is some $\varepsilon > 0$ such that

$$\forall \lambda \in]\lambda_0 - \varepsilon, \lambda_0 + \varepsilon[: E_\lambda^A = E_{\lambda_0}^A.$$

Therefore, if $\mathfrak{B} \in \mathcal{Q}(\mathcal{R})$ is in the inverse image of λ_0 by f_A , then $f_A(\mathfrak{B}) \leq \lambda_0 - \varepsilon$, a contradiction. Thus $f_A \subseteq sp(A)$.

Conversely let $\lambda_0 \in sp(A)$. There are two (non-excluding) possibilities:

- (i) There is a decreasing sequence $(\lambda_n)_{n \in \mathbb{N}}$ such that $\lambda_0 = \lim_{n \rightarrow \infty} \lambda_n$ and $E_{\lambda_{n+1}}^A < E_{\lambda_n}^A$ for all n .

- (ii) $E_\lambda^A < E_{\lambda_0}^A$ for all $\lambda < \lambda_0$.

¹We do not consider restrictions of spectral families in this and the remaining sections of this chapter, so there is no danger to confuse E^A with a restriction of some spectral family E .

If the first possibility occurs, we take a quasipoint $\mathfrak{B} \in \mathcal{Q}(\mathcal{R})$ that contains all $E_\mu^A - E_{\lambda_0}^A$ for $\mu > \lambda_0$. Then $E_\lambda^A \in \mathfrak{B}$ for all $\lambda > \lambda_0$, but $E_{\lambda_0}^A \notin \mathfrak{B}$. Hence $f_A(\mathfrak{B}) = \lambda_0$. If the second possibility occurs, we take a quasipoint \mathfrak{B} that contains all $E_{\lambda_0}^A - E_\lambda^A$ for $\lambda < \lambda_0$. Then $E_{\lambda_0}^A \in \mathfrak{B}$ but, $E_\lambda^A \notin \mathfrak{B}$ for $\lambda < \lambda_0$. Hence $f_A(\mathfrak{B}) = \lambda_0$. \square

Remark 2.5 *The foregoing proof shows that the infimum in the definition of observable functions can, in general, not be replaced by a minimum.*

Example 2.2 *The observable function of a projection $P \in \mathcal{P}(\mathcal{R})$ is given by*

$$f_P = 1 - \chi_{\mathcal{Q}_{I-P}(\mathcal{R})},$$

where $\chi_{\mathcal{Q}_{I-P}(\mathcal{R})}$ denotes the characteristic function of the open closed set $\mathcal{Q}_{I-P}(\mathcal{R})$. Hence f_P is a continuous function.

Proof: The spectral family E^P of P is

$$E_\lambda^P = \begin{cases} 0 & \text{for } \lambda < 0 \\ I - P & \text{for } 0 \leq \lambda < 1 \\ I & \text{for } 1 \leq \lambda. \end{cases}$$

The assertion follows now directly from the definition of observable functions. \square

This example can be generalized easily. Using the fact that for $a \in \mathbb{R}$ and $A \in \mathcal{R}_{sa}$

$$E^{A+aI} = E^A \circ T_a$$

holds, where T_a denotes the translation $\lambda \mapsto \lambda - a$, we obtain

Lemma 2.1 *If $A \in \mathcal{R}_{sa}$ and $a \in \mathbb{R}$, then $f_{A+aI} = a + f_A$.*

Proof: For all $\mathfrak{B} \in \mathcal{Q}(\mathcal{R})$ we have

$$\begin{aligned} f_{A+aI}(\mathfrak{B}) &= \inf\{\lambda \mid E_\lambda^{A+aI} \in \mathfrak{B}\} \\ &= \inf\{\lambda \mid E_{\lambda-a}^A \in \mathfrak{B}\} \\ &= \inf\{a + \lambda - a \mid E_{\lambda-a}^A \in \mathfrak{B}\} \\ &= a + \inf\{\lambda \mid E_\lambda^A \in \mathfrak{B}\} \\ &= a + f_A(\mathfrak{B}). \quad \square \end{aligned}$$

Consider pairwise orthogonal projections $P_1, \dots, P_n \in \mathcal{R}$, nonzero real numbers $a_1 < \dots < a_n$ and let $A := \sum_{j=1}^n a_j P_j$, $P := \sum_{j=1}^n P_j$. Then for all $a \in \mathbb{R}$

$$\begin{aligned} A - aI &= A - aP_1 - \dots - aP_n - a(I - P) \\ &= (a_1 - a)P_1 + \dots + (a_n - a)P_n + (-a)(I - P). \end{aligned}$$

Choose $a > 0$ such that $a_j - a < 0$ for all $j = 1, \dots, n$. If there is some j_0 such that $a_{j_0} < 0 < a_{j_0+1}$ then

$$a_1 - a < \dots < a_{j_0} - a < -a < a_{j_0+1} - a < \dots < a_n - a < 0.$$

(The cases when the a_k are all positive or all negative are handled analogously.) For $k = 1, \dots, n+1$ set

$$b_k := \begin{cases} a_k - a & \text{for } k \leq j_0 \\ -a & \text{for } k = j_0 + 1 \\ a_{k-1} - a & \text{for } k \geq j_0 + 2 \end{cases}$$

and

$$Q_k := \begin{cases} P_k & \text{for } k \leq j_0 \\ I - P & \text{for } k = j_0 + 1 \\ P_{k-1} & \text{for } k \geq j_0 + 2. \end{cases}$$

(Q_1, \dots, Q_{n+1}) is an orthogonal decomposition of I and therefore the spectral family of $A - aI = \sum_{k=1}^{n+1} b_k Q_k$ is given by

$$E_\lambda^{A-aI} = \begin{cases} 0 & \text{for } \lambda < b_1 \\ Q_1 + \dots + Q_k & \text{for } b_k \leq \lambda < b_{k+1} \\ I & \text{for } \lambda \geq b_{n+1}. \end{cases}$$

From the definition of observable functions we obtain

$$f_{A-aI}(\mathfrak{B}) = b_k \quad \text{for } Q_1 + \dots + Q_k \in \mathfrak{B}, \quad Q_1 + \dots + Q_{k-1} \notin \mathfrak{B}.$$

Therefore, setting $Q_0 := 0$, we get

$$f_{A-aI} = \sum_{k=1}^{n+1} b_k \chi_{\mathcal{Q}_{Q_1+\dots+Q_k}(\mathcal{R}) \setminus \mathcal{Q}_{Q_1+\dots+Q_{k-1}}(\mathcal{R})}.$$

Combining this result with lemma 2.1 gives a proof of

Proposition 2.6 *Let $P_1, P_2, \dots, P_n \in \mathcal{P}(\mathcal{R})$ be pairwise orthogonal projections and $A := \sum_{k=1}^n a_k P_k$ with real coefficients a_1, a_2, \dots, a_n . Then, setting $P_0 := 0$,*

$$f_A = \sum_{k=1}^n a_k \chi_{\mathcal{Q}_{P_1+\dots+P_k}(\mathcal{R}) \setminus \mathcal{Q}_{P_1+\dots+P_{k-1}}(\mathcal{R})}.$$

Consequently, f_A is continuous.

Corollary 2.1 *If \mathcal{R} is abelian and $A = \sum_{k=1}^n a_k P_k$ as in proposition 2.6, then*

$$f_A = \sum_{k=1}^n a_k \chi_{\mathcal{Q}_{P_k}(\mathcal{R})}.$$

Proof: If \mathcal{R} is abelian then the projection lattice $\mathcal{P}(\mathcal{R})$ is distributive and therefore

$$\mathcal{Q}_{P_1+\dots+P_k}(\mathcal{R}) = \bigcup_{j=1}^k \mathcal{Q}_{P_j}(\mathcal{R}). \quad \square$$

Theorem 2.4 *Let $A \in \mathcal{R}_{sa}$. Then the observable function $f_A : \mathcal{Q}(\mathcal{R}) \rightarrow \mathbb{R}$ is continuous.*

Proof: We know from theorem 2.3 that the image of f_A is the spectrum $sp(A)$ of A . Let $a := \min(sp(A))$, $b := \max(sp(A))$ and let $\varepsilon > 0$. Choose $\lambda_0 \in]a - \varepsilon, a[$, $\lambda_n \in]b, b + \varepsilon[$, $\lambda_1, \dots, \lambda_{n-1} \in [a, b]$ such that $\lambda_{k-1} < \lambda_k$ and $\lambda_k - \lambda_{k-1} < \varepsilon$ for $k = 1, \dots, n$. Moreover let $\lambda_k^* \in]\lambda_{k-1}, \lambda_k[$ for $k = 1, \dots, n$ and

$$A_\varepsilon := \sum_{k=1}^n \lambda_k^* (E_{\lambda_k}^A - E_{\lambda_{k-1}}^A) := \sum_{k=1}^n \lambda_k^* P_k$$

with $P_k := E_{\lambda_k}^A - E_{\lambda_{k-1}}^A$. Then by proposition 2.6

$$f_{A_\varepsilon} = \sum_{k=1}^n \lambda_k^* \chi_{\mathcal{Q}_{P_1+\dots+P_k}(\mathcal{R}) \setminus \mathcal{Q}_{P_1+\dots+P_{k-1}}(\mathcal{R})} = \sum_{k=1}^n \lambda_k^* \chi_{\mathcal{Q}_{E_{\lambda_k}^A}(\mathcal{R}) \setminus \mathcal{Q}_{E_{\lambda_{k-1}}^A}(\mathcal{R})}.$$

Let $\mathfrak{B} \in \mathcal{Q}(\mathcal{R})$. Then $\mathfrak{B} \in \mathcal{Q}_{E_{\lambda_k}^A}(\mathcal{R}) \setminus \mathcal{Q}_{E_{\lambda_{k-1}}^A}(\mathcal{R})$ for exactly one k . Hence

$$f_{A_\varepsilon}(\mathfrak{B}) = \lambda_k^*$$

and

$$f_A(\mathfrak{B}) = \inf\{\lambda \mid E_\lambda^A \in \mathfrak{B}\} \in [\lambda_{k-1}, \lambda_k].$$

This implies

$$|f_A(\mathfrak{B}) - f_{A_\varepsilon}(\mathfrak{B})| < \varepsilon$$

and, \mathfrak{B} being arbitrary,

$$\|f_A - f_{A_\varepsilon}\|_\infty \leq \varepsilon.$$

Hence f_A is continuous. \square

Definition 2.4 Let \mathcal{R} be a von Neumann algebra. Then we denote by $\mathcal{O}(\mathcal{R})$ the set of observable functions $\mathcal{Q}(\mathcal{R}) \rightarrow \mathbb{R}$.

By the foregoing result, $\mathcal{O}(\mathcal{R})$ is a subset of $C_b(\mathcal{Q}(\mathcal{R}), \mathbb{R})$, the algebra of all bounded continuous functions $\mathcal{Q}(\mathcal{R}) \rightarrow \mathbb{R}$. $\mathcal{O}(\mathcal{R})$ separates the points of $\mathcal{Q}(\mathcal{R})$ because the observable function of a projection P is $f_P = 1 - \chi_{\mathcal{Q}_{I-P}(\mathcal{R})}$. Moreover, it contains the constant functions (by lemma 2.1). In general, however, it is not an algebra and not even a vector space (with respect to the pointwise defined algebraic operations).

Theorem 2.5 Let \mathcal{R} be a von Neumann algebra and let $\mathcal{O}(\mathcal{R})$ be the set of observable functions on $\mathcal{Q}(\mathcal{R})$. Then

$$\mathcal{O}(\mathcal{R}) = C_b(\mathcal{Q}(\mathcal{R}), \mathbb{R})$$

if and only if \mathcal{R} is abelian.

We will prove here only one half of this theorem leaving the other one until we have proved an abstract characterization of observable functions.

Proposition 2.7 Let \mathcal{R} be a von Neumann algebra such that for all $P \in \mathcal{P}(\mathcal{R})$ the characteristic function $\chi_{\mathcal{Q}_P(\mathcal{R})}$ of the open closed set $\mathcal{Q}_P(\mathcal{R})$ is an observable function. Then \mathcal{R} is abelian.

Proof: Let $P_0 \in \mathcal{P}(\mathcal{R})$. By assumption $\chi_{\mathcal{Q}_{P_0}(\mathcal{R})}$ is the observable function f of an element $A \in \mathcal{R}_{sa}$. Then $sp(A) = im f \subseteq \{0, 1\}$ and therefore A is a projection P in \mathcal{R} . Hence

$$\chi_{\mathcal{Q}_{P_0}(\mathcal{R})} = 1 - \chi_{\mathcal{Q}_{I-P}(\mathcal{R})}. \quad (2.1)$$

Let $\mathfrak{B} \in \mathcal{Q}_P(\mathcal{R})$. Then $\chi_{\mathcal{Q}_{P_0}(\mathcal{R})}(\mathfrak{B}) = 1$ and therefore $P_0 \in \mathfrak{B}$. Hence we have shown

$$\forall \mathfrak{B} \in \mathcal{Q}(\mathcal{R}) : (P \in \mathfrak{B} \implies P_0 \in \mathfrak{B}). \quad (2.2)$$

We show that this implies

$$P = P_0. \quad (2.3)$$

This is equivalent to

$$1 = \chi_{\mathcal{Q}_P(\mathcal{R})} + \chi_{\mathcal{Q}_{I-P}(\mathcal{R})} \quad (2.4)$$

i.e. to

$$\mathcal{Q}(\mathcal{R}) = \mathcal{Q}_P(\mathcal{R}) \cup \mathcal{Q}_{I-P}(\mathcal{R}). \quad (2.5)$$

By proposition 3.5 in part I ([7]), this property is equivalent to the distributivity of the lattice $\mathcal{P}(\mathcal{R})$, i.e. to the commutativity of \mathcal{R} .

Assume that 2.3 does not hold, i.e. $P_0 \wedge P < P$ or $P < P_0$. If $P_0 \wedge P < P$, take some $\mathfrak{B} \in \mathcal{Q}(\mathcal{R})$ with $P - P_0 \wedge P \in \mathfrak{B}$. Then $P \in \mathfrak{B}$, hence also $P_0 \in \mathfrak{B}$ (by 2.2) and therefore $P_0 \wedge P \in \mathfrak{B}$, a contradiction. This shows $P_0 \wedge P = P$ i.e. $P \leq P_0$. Assume that $P < P_0$. Let $\mathfrak{B} \in \mathcal{Q}(\mathcal{R})$ such that $P_0 - P \in \mathfrak{B}$. Then $P_0 \in \mathfrak{B}$ and therefore 2.1 implies $I - P \notin \mathfrak{B}$. But then

$$P_0 - P = P_0(I - P) = P_0 \wedge (I - P) \notin \mathfrak{B},$$

a contradiction again. Hence 2.3 holds. \square

The foregoing proposition can be reformulated in the following way.

If $E = (E_\lambda)_{\lambda \in \mathbb{R}}$ is a bounded (right-continuous) spectral family, a natural candidate for an orthocomplement of E in the lattice of bounded spectral families ([6]) would be

$$\tilde{E} : \lambda \mapsto I - E_{-\lambda}.$$

But \tilde{E} is, continuous from the left and, in general, not from the right. Spectralization of \tilde{E} gives the (right-continuous) spectral family

$$\begin{aligned} (\neg E)_\lambda &:= \bigwedge_{\mu > \lambda} (I - E_{-\mu}) \\ &= I - \bigvee_{\mu < -\lambda} E_\mu \\ &= I - E_{-\lambda-}. \end{aligned}$$

A routine calculation shows ([6]) that, if $A \in \mathcal{R}_{sa}$ is the operator corresponding to E , then the operator corresponding to $\neg E$ is $-A$. Therefore, we obtain for the negative of the observable function of $-A$:

$$\begin{aligned} -f_{-A}(\mathfrak{B}) &= -\inf\{\lambda \in \mathbb{R} \mid I - E_{-\lambda-} \in \mathfrak{B}\} \\ &= -\inf\{-\lambda \in \mathbb{R} \mid I - E_{\lambda-} \in \mathfrak{B}\} \\ &= \sup\{\lambda \in \mathbb{R} \mid I - E_{\lambda-} \in \mathfrak{B}\} \\ &= \sup\{\lambda \in \mathbb{R} \mid I - E_\lambda \in \mathfrak{B}\}. \end{aligned}$$

In particular, for a projection P we get

$$-f_{-P} = \chi_{\mathcal{Q}_P(\mathcal{R})}.$$

Corollary 2.2 *A von Neumann algebra \mathcal{R} is abelian if and only if*

$$\mathcal{O}(\mathcal{R}) = -\mathcal{O}(\mathcal{R}).$$

Proof: If \mathcal{R} is abelian and $\mathfrak{B} \in \mathcal{Q}(\mathcal{R})$, then $P \in \mathfrak{B}$ or $I - P \in \mathfrak{B}$ for all $P \in \mathcal{P}(\mathcal{R})$. Hence

$$-f_{-A}(\mathfrak{B}) = \sup\{\lambda \in \mathbb{R} \mid I - E_\lambda^A \in \mathfrak{B}\} = \inf\{\lambda \in \mathbb{R} \mid E_\lambda^A \in \mathfrak{B}\} = f_A(\mathfrak{B})$$

for all $A \in \mathcal{R}_{sa}$ and all $\mathfrak{B} \in \mathcal{Q}(\mathcal{R})$. \square

Remark 2.6 *The functions*

$$\begin{aligned} g_A : \mathcal{Q}(\mathcal{R}) &\rightarrow \mathbb{R} \\ \mathfrak{B} &\mapsto \sup\{\lambda \in \mathbb{R} \mid I - E_\lambda^A \in \mathfrak{B}\} \end{aligned}$$

were introduced by Döring ([13]). His motivation came from the following observation. Let $A \in \mathcal{L}(\mathcal{H})_{sa}$ and $x \in \mathcal{H} \setminus \{0\}$. Then, by the spectral theorem, we have

$$\langle Ax, x \rangle = \int_{-|A|}^{|A|} \lambda d \langle E_\lambda^A x, x \rangle.$$

It is obvious that

$$\langle E_\lambda^A x, x \rangle = \begin{cases} \langle x, x \rangle & \text{for } \lambda > f_A(\mathfrak{B}_{\mathbb{C}x}) \\ 0 & \text{for } \lambda < g_A(\mathfrak{B}_{\mathbb{C}x}), \end{cases}$$

where $\mathfrak{B}_{\mathbb{C}x}$ is the atomic quasipoint defined by $P_{\mathbb{C}x}$. Hence

$$\langle Ax, x \rangle = \int_{g_A(\mathfrak{B}_{\mathbb{C}x})}^{f_A(\mathfrak{B}_{\mathbb{C}x})} \lambda d \langle E_\lambda^A x, x \rangle.$$

Döring called the function $g_A : \mathcal{Q}(\mathcal{R}) \rightarrow \mathbb{R}$ the **antonymous function** induced by $A \in \mathcal{R}_{sa}$. Because of $g_A = -f_{-A}$ for all $A \in \mathcal{R}_{sa}$, the set of antonymous functions is simply $-\mathcal{O}(\mathcal{R})$. Therefore, we prefer the name **mirrored observable function** instead of the pretentious “antonymous function”. Of course, the properties of mirrored observable functions are quite analogous to those of observable functions. For example, we have

$$\text{im}(-f_{-A}) = -\text{im}f_{-A} = -\text{sp}(-A) = \text{sp}(A).$$

Similarly, it is quite easy to translate the whole discussion of the next sections to the mirrored case. Observable and mirrored observable functions are two sides of the same coin. Their symmetric rôle will become more clear in part IV ([9]).

2.3 Abstract Characterization of Observable Functions

We will prove some properties of observable functions of a von Neumann algebra \mathcal{R} which in turn will serve as axioms for an abstract notion of observable function.

Definition 2.5 *Let \mathbb{L} be a complete lattice (with minimal element 0 and maximal element 1). A nonempty subset $\mathcal{J} \subseteq \mathbb{L}$ is called a **dual ideal** if it has the following properties:*

- (i) $0 \notin \mathcal{J}$,
- (ii) $a, b \in \mathcal{J} \implies a \wedge b \in \mathcal{J}$,
- (iii) if $a \in \mathcal{J}$ and $a \leq b$, then $b \in \mathcal{J}$.

If $a \in \mathbb{L} \setminus \{0\}$, the dual ideal

$$H_a := \{b \in \mathbb{L} \mid b \geq a\}$$

is called the **principal dual ideal generated by a** . We denote by $\mathcal{D}(\mathbb{L})$ the set of dual ideals of \mathbb{L} . For $a \in \mathbb{L}$ let

$$\mathcal{D}_a(\mathbb{L}) := \{\mathcal{J} \in \mathcal{D}(\mathbb{L}) \mid a \in \mathcal{J}\}.$$

As mentioned earlier, a maximal dual ideal is nothing but a quasipoint of \mathbb{L} . We collect some obvious properties of the sets $\mathcal{D}_a(\mathbb{L})$ in the following

Remark 2.7 *For all $a, b \in \mathbb{L}$ the following properties hold:*

- (i) $a \leq b \implies \mathcal{D}_a(\mathbb{L}) \subseteq \mathcal{D}_b(\mathbb{L})$,
- (ii) $\mathcal{D}_{a \wedge b}(\mathbb{L}) = \mathcal{D}_a(\mathbb{L}) \cap \mathcal{D}_b(\mathbb{L})$,
- (iii) $\mathcal{D}_a(\mathbb{L}) \cup \mathcal{D}_b(\mathbb{L}) \subseteq \mathcal{D}_{a \vee b}(\mathbb{L})$,
- (iv) $\mathcal{D}_0(\mathbb{L}) = \emptyset$, $\mathcal{D}_1(\mathbb{L}) = \mathcal{D}(\mathbb{L})$.

These properties show in particular that $\{\mathcal{D}_a(\mathbb{L}) \mid a \in \mathbb{L}\}$ is a basis of a topology on $\mathcal{D}(\mathbb{L})$. The Stone spectrum $\mathcal{Q}(\mathbb{L})$ is dense in $\mathcal{D}(\mathbb{L})$ with respect to this topology.

Note that $\mathcal{D}(\mathbb{L})$ is in general **not a Hausdorff space**: let $b, c \in \mathbb{L}, b < c$. Then $H_c \in \mathcal{D}_a(\mathbb{L}) \iff c \leq a$, hence $H_b \in \mathcal{D}_a(\mathbb{L})$ and, therefore, H_b and H_c cannot be separated. We return to the case that \mathbb{L} is the projection lattice $\mathcal{P}(\mathcal{R})$ of a von Neumann algebra \mathcal{R} , although most of our considerations also hold for an arbitrary orthocomplemented complete lattice.

Lemma 2.2 $\forall P \in \mathcal{P}(\mathcal{R}) : H_P = \bigcap_{P \in \mathfrak{B}} \mathfrak{B} \quad (=: \bigcap \mathcal{Q}_P(\mathcal{R})).$

Proof: If $P \in \mathfrak{B}$ then clearly $H_P \subseteq \mathfrak{B}$. Hence $H_P \subseteq \bigcap \mathcal{Q}_P(\mathcal{R})$. Conversely, assume that $Q \in \bigcap \mathcal{Q}_P(\mathcal{R}) \setminus H_P$. Then $P \wedge Q < P$ and so there is some quasipoint \mathfrak{B} which contains $P - P \wedge Q$. But then $P \in \mathfrak{B}$ and therefore also $P \wedge Q \in \mathfrak{B}$, a contradiction. \square

Let $A \in \mathcal{R}_{sa}$ with corresponding spectral family E^A and observable function f_A . We extend f_A to a function $\mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ on the space $\mathcal{D}(\mathcal{R})$ of dual ideals of $\mathcal{P}(\mathcal{R})$ (and we denote this extension again by f_A) in a natural manner:

$$\forall \mathcal{J} \in \mathcal{D}(\mathcal{R}) : f_A(\mathcal{J}) := \inf\{\lambda \mid E_\lambda^A \in \mathcal{J}\}.$$

Proposition 2.8 *Let $(\mathcal{J}_j)_{j \in J}$ be a family in $\mathcal{D}(\mathcal{R})$. Then*

$$f_A\left(\bigcap_{j \in J} \mathcal{J}_j\right) = \sup_{j \in J} f_A(\mathcal{J}_j).$$

Proof: $\mathcal{J} := \bigcap_{j \in J} \mathcal{J}_j$ is a dual ideal that is contained in \mathcal{J}_j for all $j \in J$. Hence

$$f_A(\mathcal{J}_j) = \inf\{\lambda \mid E_\lambda^A \in \mathcal{J}_j\} \leq \inf\{\lambda \mid E_\lambda^A \in \mathcal{J}\} = f_A(\mathcal{J})$$

and therefore

$$\sup_j f_A(\mathcal{J}_j) \leq f_A(\mathcal{J}).$$

Let $\varepsilon > 0$ and choose λ such that

$$f_A(\mathcal{J}) - \varepsilon < \lambda < f_A(\mathcal{J}).$$

Then $E_\lambda^A \notin \mathcal{J}$, so there is some j_0 such that $E_\lambda^A \notin \mathcal{J}_{j_0}$. This means $f_A(\mathcal{J}_{j_0}) \geq \lambda$ and therefore we obtain

$$f_A(\mathcal{J}) - \varepsilon < f_A(\mathcal{J}_{j_0}) \leq \sup_j f_A(\mathcal{J}_j).$$

As $\varepsilon > 0$ was arbitrary, we conclude

$$f_A(\mathcal{J}) \leq \sup_j f_A(\mathcal{J}_j). \quad \square$$

On a principal dual ideal H_P we simply have

$$f_A(H_P) = \inf\{\lambda \mid E_\lambda^A \geq P\} = \min\{\lambda \mid E_\lambda^A \geq P\}$$

because E^A is continuous from the right.

The following characterization of eigenvalues of selfadjoint operators is an application of the foregoing results. It is a generalization of 5.7.22 in [20] for the selfadjoint case. In particular, it gives a further interesting characterization of finite von Neumann algebras.

Proposition 2.9 *Let $A \in \mathcal{R}_{sa}$ and $\lambda \in sp(A)$. If λ is an eigenvalue, the interior of $f_A(\lambda)$ is a nonvoid subset of $\mathcal{Q}(\mathcal{R})$. The converse holds if and only if \mathcal{R} is a finite von Neumann algebra.*

Proof: λ is an eigenvalue of A if and only if $E_\lambda^A - E_{\lambda-}^A > 0$. Take $\mathfrak{B} \in \mathcal{Q}_{E_\lambda^A - E_{\lambda-}^A}(\mathcal{R})$. Then $E_\lambda^A \in \mathfrak{B}$, hence $f_A(\mathfrak{B}) \leq \lambda$. If $f_A(\mathfrak{B}) < \lambda$, then $E_\mu^A \in \mathfrak{B}$ for some $\mu < \lambda$. But this implies $E_{\lambda-}^A \in \mathfrak{B}$, a contradiction. Hence $f_A(\mathfrak{B}) = \lambda$ and, therefore, $\mathcal{Q}_{E_\lambda^A - E_{\lambda-}^A}(\mathcal{R}) \subseteq f_A(\lambda)$.

Let \mathcal{R} be finite and let $\lambda \in sp(A)$ such that $\text{int} f_A(\lambda) \neq \emptyset$. Then there is some $P \in \mathcal{P}_0(\mathcal{R})$ with $\mathcal{Q}_P(\mathcal{R}) \subseteq f_A(\lambda)$. Since $H_P = \bigcap \mathcal{Q}_P(\mathcal{R})$ by lemma 2.2, we obtain

$$f_A(H_P) = \sup_{\mathfrak{B} \in \mathcal{Q}_P(\mathcal{R})} f_A(\mathfrak{B}) = \lambda.$$

This implies

$$E_\lambda^A \geq P.$$

But then

$$\forall \mu < \lambda : E_\mu^A \wedge P = 0,$$

because, if $E_\mu^A \wedge P \neq 0$ for some $\mu < \lambda$, there is $\mathfrak{B} \in \mathcal{Q}_P(\mathcal{R})$ with $E_\mu^A \wedge P \in \mathfrak{B}$, hence $E_\mu^A \in \mathfrak{B}$ and therefore $f_A(\mathfrak{B}) \leq \mu < \lambda$, a contradiction. Now assume that $E_\lambda^A = E_{\lambda-}^A$. From the finiteness of \mathcal{R} we conclude

$$0 = \bigvee_{\mu < \lambda} (P \wedge E_\mu^A) = P \wedge \bigvee_{\mu < \lambda} E_\mu^A = P \wedge E_\lambda^A = P,$$

a contradiction.

If \mathcal{R} is not finite, we have to present an operator $A \in \mathcal{R}_{sa}$ that has a spectral

value $\lambda \in sp(A)$ such that $\text{int}f_A^{-1}(\lambda) \neq \emptyset$ and λ is not an eigenvalue. Assume that \mathcal{R} is not finite. Then \mathcal{R} contains a direct summand of the form $\mathcal{M} \bar{\otimes} \mathcal{L}(\mathcal{H}_0)$, where $\mathcal{M} \subseteq \mathcal{L}(\mathcal{K})$ is a suitable von Neumann algebra and \mathcal{H}_0 a separable Hilbert space of infinite dimension (see the proof of theorem 3.1 in [7]). So it suffices to find an example in $\mathcal{M} \bar{\otimes} \mathcal{L}(\mathcal{H}_0)$.

Let $(e_n)_{n \in \mathbb{N}}$ be an orthonormal basis of \mathcal{H}_0 and for $\lambda \in \mathbb{R}$ let

$$E_\lambda := \begin{cases} 0 & \text{for } \lambda < 0 \\ \sum_{n=1}^k P_{\mathbb{C}e_n} & \text{for } 1 - \frac{1}{k} \leq \lambda < 1 - \frac{1}{k+1} \text{ and } k \in \mathbb{N} \\ I & \text{for } \lambda \geq 1. \end{cases}$$

$(E_\lambda)_{\lambda \in \mathbb{R}}$ is a bounded spectral family and therefore $(I_{\mathcal{M}} \otimes E_\lambda)_{\lambda \in \mathbb{R}}$ is a spectral family in $\mathcal{M} \bar{\otimes} \mathcal{L}(\mathcal{H}_0)$. 1 belongs to the spectrum of the corresponding selfadjoint operator $I_{\mathcal{M}} \otimes A \in \mathcal{M} \bar{\otimes} \mathcal{L}(\mathcal{H}_0)$ but, by construction, 1 is not an eigenvalue of $I_{\mathcal{M}} \otimes A$. We have

$$\mathcal{R} = \mathcal{M} \bar{\otimes} \mathcal{L}(\mathcal{H}_0) \oplus \mathcal{S}$$

with a suitable von Neumann subalgebra \mathcal{S} of \mathcal{R} . It is then easy to see that a quasipoint in $\mathcal{P}(\mathcal{R})$ is either of the form $\mathfrak{B} \oplus \mathcal{P}(\mathcal{S})$, with $\mathfrak{B} \in \mathcal{Q}(\mathcal{M} \bar{\otimes} \mathcal{L}(\mathcal{H}_0))$, or it is of the form $\mathcal{P}(\mathcal{M} \bar{\otimes} \mathcal{L}(\mathcal{H}_0)) \oplus \mathfrak{C}$ with $\mathfrak{C} \in \mathcal{Q}(\mathcal{S})$. Now let $x := \sum_{n=1}^{\infty} \frac{1}{n} e_n$. Then the quasipoints that contain $(I_{\mathcal{M}} \otimes P_{\mathbb{C}x}, 0)$ are of the form $\mathfrak{B} \oplus \mathcal{P}(\mathcal{S})$. Thus we can restrict our considerations to quasipoints $\mathfrak{B} \in \mathcal{Q}(\mathcal{M} \bar{\otimes} \mathcal{L}(\mathcal{H}_0))$ that contain $I_{\mathcal{M}} \otimes P_{\mathbb{C}x}$. If \mathfrak{B} is such a quasipoint, then $I_{\mathcal{M}} \otimes E_\lambda \notin \mathfrak{B}$ for $\lambda < 1$, since $E_\lambda \wedge P_{\mathbb{C}x} = 0$. Hence

$$f_{I_{\mathcal{M}} \otimes A}(\mathfrak{B}) = 1,$$

and this implies that the open set $\mathcal{Q}_{I_{\mathcal{M}} \otimes P_{\mathbb{C}x}}(\mathcal{M} \bar{\otimes} \mathcal{L}(\mathcal{H}_0))$ is contained in $f_{I_{\mathcal{M}} \otimes A}^{-1}(1)$. \square

We proceed with the development of the general theory.

Proposition 2.10 $f_A : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ is upper semicontinuous.

Proof: We have to show that the following property holds:

$$\forall \mathcal{J}_0 \in \mathcal{D}(\mathcal{R}) \forall \varepsilon > 0 \exists P \in \mathcal{J}_0 \forall \mathcal{J} \in \mathcal{D}_P(\mathcal{R}) : f_A(\mathcal{J}) < f_A(\mathcal{J}_0) + \varepsilon.$$

Indeed $P := E_{f_A(\mathcal{J}_0) + \frac{\varepsilon}{2}}^A \in \mathcal{J}_0$ and therefore

$$f_A(\mathcal{J}) \leq f_A(\mathcal{J}_0) + \frac{\varepsilon}{2} < f_A(\mathcal{J}_0) + \varepsilon$$

for all $\mathcal{J} \in \mathcal{D}_P(\mathcal{R})$. \square

Remark 2.8 *Observable functions $f_A : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ are not continuous in general.*

Proof: The observable function $f_P : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ of a projection $P \in \mathcal{R}$ is given by

$$f_P = 1 - \chi_{\mathcal{D}_P(\mathcal{R})}.$$

f_P is continuous if and only if $\mathcal{D}_P(\mathcal{R})$ is open (which is true by definition) and closed. Now

$$\begin{aligned} \mathcal{J} \in \overline{\mathcal{D}_P(\mathcal{R})} &\iff \forall Q \in \mathcal{J} : \mathcal{D}_Q(\mathcal{R}) \cap \mathcal{D}_P(\mathcal{R}) \neq \emptyset \\ &\iff \forall Q \in \mathcal{J} : \mathcal{D}_{P \wedge Q}(\mathcal{R}) \neq \emptyset \\ &\iff \forall Q \in \mathcal{J} : P \wedge Q \neq 0 \end{aligned}$$

and therefore $\overline{\mathcal{D}_P(\mathcal{R})} = \mathcal{D}_P(\mathcal{R})$ if and only if

$$\forall \mathcal{J} \in \mathcal{D}(\mathcal{R}) : ((\forall Q \in \mathcal{J} : P \wedge Q \neq 0) \implies P \in \mathcal{J}).$$

This leads to the following example. Let $P, P_1 \in \mathcal{P}(\mathcal{R})$ such that $0 \neq P < P_1$. Then $Q \wedge P = P$ for all $Q \in H_{P_1}$ but $P \notin H_{P_1}$. Hence $H_{P_1} \in \overline{\mathcal{D}_P(\mathcal{R})} \setminus \mathcal{D}_P(\mathcal{R})$. \square

Proposition 2.11 *For any function $f : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$, the following two properties are equivalent:*

- (i) f is upper semicontinuous and decreasing (i.e. $\mathcal{J}_1 \subseteq \mathcal{J}_2 \implies f(\mathcal{J}_2) \leq f(\mathcal{J}_1)$).
- (ii) $\forall \mathcal{J} \in \mathcal{D}(\mathcal{R}) : f(\mathcal{J}) = \inf\{f(H_P) \mid P \in \mathcal{J}\}$.

Proof: Assume that (i) holds. Let $\mathcal{J}_0 \in \mathcal{D}(\mathcal{R})$ and $P \in \mathcal{J}_0$. Then $f(\mathcal{J}_0) \leq f(H_P)$ and therefore $f(\mathcal{J}_0) \leq \inf\{f(H_P) \mid P \in \mathcal{J}_0\}$. Let $\varepsilon > 0$. Then by the upper semicontinuity of f

$$\exists P_0 \in \mathcal{J}_0 \forall \mathcal{J} \in \mathcal{D}_{P_0}(\mathcal{R}) : f(\mathcal{J}) < f(\mathcal{J}_0) + \varepsilon,$$

in particular

$$f(H_{P_0}) < f(\mathcal{J}_0) + \varepsilon.$$

Hence $f(\mathcal{J}_0) = \inf\{f(H_P) \mid P \in \mathcal{J}_0\}$.

Conversely assume that (ii) holds. Let $\mathcal{J}_1, \mathcal{J}_2 \in \mathcal{D}(\mathcal{R})$, $\mathcal{J}_1 \subseteq \mathcal{J}_2$. Then

$$f(\mathcal{J}_2) = \inf\{f(H_P) \mid P \in \mathcal{J}_2\} \leq \inf\{f(H_P) \mid P \in \mathcal{J}_1\} = f(\mathcal{J}_1),$$

i.e. f is decreasing. Let $\mathcal{J}_0 \in \mathcal{D}(\mathcal{R})$ and $\varepsilon > 0$. There is some $P \in \mathcal{J}_0$ such that $f(H_P) < f(\mathcal{J}_0) + \varepsilon$. If $\mathcal{J} \in \mathcal{D}_P(\mathcal{R})$ then $H_P \subseteq \mathcal{J}$ and therefore

$$f(\mathcal{J}) \leq f(H_P) < f(\mathcal{J}_0) + \varepsilon.$$

\square

Corollary 2.3 $f_A(\mathcal{D}(\mathcal{R})) = sp(A)$ for all $A \in \mathcal{R}_{sa}$.

Proof: We already know that $f_A(\mathcal{Q}(\mathcal{R})) = sp(A)$, so it suffices to prove that $f_A(\mathcal{D}(\mathcal{R}))$ is contained in $sp(A)$. But this follows from propositions 2.8, 2.10, 2.11 and the closedness of $sp(A)$:

$$\forall \mathcal{J} \in \mathcal{D}(\mathcal{R}) : f_A(\mathcal{J}) = \inf_{P \in \mathcal{J}} \sup_{\mathfrak{B} \in \mathcal{Q}_P(\mathcal{R})} f_A(\mathfrak{B}) \in sp(A). \quad \square$$

Definition 2.6 A function $f : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ is called an **abstract observable function** if it is upper semicontinuous and satisfies the intersection condition

$$f\left(\bigcap_{j \in J} \mathcal{J}_j\right) = \sup_{j \in J} f(\mathcal{J}_j)$$

for all families $(\mathcal{J}_j)_{j \in J}$ in $\mathcal{D}(\mathcal{R})$.

The intersection condition implies that an abstract observable function is decreasing. Hence by 2.11 the definition of abstract observable functions can be reformulated as follows:

Remark 2.9 $f : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ is an observable function if and only if the following two properties hold for f :

- (i) $\forall \mathcal{J} \in \mathcal{D}(\mathcal{R}) : f(\mathcal{J}) = \inf\{f(H_P) \mid P \in \mathcal{J}\},$
- (ii) $f(\bigcap_{j \in J} \mathcal{J}_j) = \sup_{j \in J} f(\mathcal{J}_j)$ for all families $(\mathcal{J}_j)_{j \in J}$ in $\mathcal{D}(\mathcal{R})$.

A direct consequence of the intersection condition is the following

Remark 2.10 Let $\lambda \in im f$. Then the inverse image $f^{-1}(\lambda) \subseteq \mathcal{D}(\mathcal{R})$ has a minimal element \mathcal{J}_λ which is simply given by

$$\mathcal{J}_\lambda = \bigcap \{\mathcal{J} \in \mathcal{D}(\mathcal{R}) \mid f(\mathcal{J}) = \lambda\}.$$

We will now show how one can recover the spectral family E^A of $A \in \mathcal{R}_{sa}$ from the observable function f_A . This gives us the decisive hint for the proof that to each abstract observable function $f : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ there is a unique $A \in \mathcal{R}_{sa}$ with $f = f_A$.

Lemma 2.3 Let $f_A : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ be an observable function and let E^A be the spectral family corresponding to A . If $\lambda \in im f$, then

$$\mathcal{J}_\lambda = \{P \in \mathcal{P}(\mathcal{R}) \mid \exists \mu > \lambda : P \geq E_\mu^A\}.$$

$\mathcal{J}_\lambda = H_{E_\lambda^A}$ if and only if E^A is constant on some interval $[\lambda, \lambda + \delta]$. Moreover

$$E_\lambda^A = \inf \mathcal{J}_\lambda.$$

Proof: It is obvious that $\mathcal{I} := \{P \in \mathcal{D}(\mathcal{R}) \mid \exists \mu > \lambda : P \geq E_\mu^A\}$ is a dual ideal and that $f_A(\mathcal{I}) = \lambda$. Let \mathcal{J} be any dual ideal in $\mathcal{D}(\mathcal{R})$ such that $f_A(\mathcal{J}) = \lambda$. Then $\lambda = \inf\{\mu \mid E_\mu^A \in \mathcal{J}\}$. For $P \in \mathcal{I}$ let $\mu > \lambda$ such that $P \geq E_\mu^A$. Then $E_\mu^A \in \mathcal{J}$ and therefore $P \in \mathcal{J}$. This shows $\mathcal{I} \subseteq \mathcal{J}$. Hence $\mathcal{J}_\lambda = \mathcal{I}$. Clearly $\mathcal{J}_\lambda = H_{E_\lambda^A}$ if and only if E^A is constant on some interval $[\lambda, \lambda + \delta]$. The last assertion is due to the continuity of E^A from the right. \square

Let $\lambda_0 \in \text{im} f_A$. Then

$$f_A(H_{E_{\lambda_0}^A}) = \lambda_0$$

if and only if there is no $\delta > 0$ such that E^A is constant on the interval $[\lambda_0 - \delta, \lambda_0]$.

Proposition 2.12 *Let $\lambda_0 \in \text{im} f_A$. Then $\lambda_0 = f_A(H)$ for some principal dual ideal $H \in \mathcal{D}(\mathcal{R})$ if and only if there is no $\delta > 0$ such that E^A is constant on the interval $[\lambda_0 - \delta, \lambda_0]$.*

Proof: Assume that E^A is constant on some interval $[\lambda_1, \lambda_0]$ with $\lambda_1 < \lambda_0$ but that there is a $P \in \mathcal{P}_0(\mathcal{R})$ such that $f_A(H_P) = \lambda_0$. Then $\mathcal{J}_{\lambda_0} \subseteq H_P$ and therefore, by lemma 2.3, $E_{\lambda_0}^A \geq P$, i.e. $H_{E_{\lambda_0}^A} \subseteq H_P$. This implies $f_A(H_{E_{\lambda_0}^A}) \geq f_A(H_P) = \lambda_0$, contradicting $f_A(H_{E_{\lambda_0}^A}) \leq \lambda_1 < \lambda_0$. \square

Corollary 2.4 *If $\lambda \in f_A(\mathcal{D}_{pr}(\mathcal{R}))$ then*

$$E_\lambda^A = \bigvee \{P \in \mathcal{P}_0(\mathcal{R}) \mid f_A(H_P) = \lambda\}.$$

Proof: Note that, by proposition 2.12, the case $E_\lambda^A = 0$ cannot occur. If $f_A(H_P) = \lambda$ then $\mathcal{J}_\lambda \subseteq H_P$ and therefore $E_\lambda^A \geq P$. Thus $\lambda = f_A(H_{E_\lambda^A}) \geq f_A(H_P) = \lambda$. \square

Corollary 2.5 *An observable function $f_A : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ is uniquely determined by its restriction to $\mathcal{Q}(\mathcal{R})$.*

Proof: By proposition 2.11 f_A is determined by its values on principle dual ideals H_P ($P \in \mathcal{P}(\mathcal{R})$). $H_P = \bigcap \mathcal{Q}_P(\mathcal{R})$ by lemma 2.2 and therefore $f_A(H_P) = \sup\{f_A(\mathfrak{B}) \mid \mathfrak{B} \in \mathcal{Q}_P(\mathcal{R})\}$ by proposition 2.8. \square

Remark 2.11 *A selfadjoint operator $A \in \mathcal{R}$ is uniquely determined by its observable function f_A .*

Proof: If $\lambda \in \text{im} f_A$ then $E_\lambda^A = \inf \mathcal{J}_\lambda$ and \mathcal{J}_λ is the minimal element of $f_A^{-1}(\lambda)$. Hence the uniqueness of A follows from the uniqueness of the spectral resolution. \square

Theorem 2.6 *Let $f : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ be an abstract observable function. Then there is a unique $A \in \mathcal{R}_{sa}$ such that $f = f_A$.*

The **proof** will proceed in three steps. In the first step we construct from the abstract observable function f an increasing family $(E_\lambda)_{\lambda \in \text{im} f}$ in $\mathcal{P}(\mathcal{R})$ and show in a second step that this family can be extended to a spectral family in \mathcal{R} . Finally, in the third step, we show that the selfadjoint operator $A \in \mathcal{R}$ corresponding to that spectral family has observable function $f_A = f$ and that A is uniquely determined by f .

Step 1 Let $\lambda \in \text{im} f$ and let $\mathcal{J}_\lambda \in \mathcal{D}(\mathcal{R})$ be the smallest dual ideal such that $f(\mathcal{J}_\lambda) = \lambda$. In view of lemma 2.3 we have no choice than to define

$$E_\lambda := \inf \mathcal{J}_\lambda.$$

Lemma 2.4 *The family $(E_\lambda)_{\lambda \in \text{im} f}$ is increasing.*

Proof: Let $\lambda, \mu \in \text{im} f$, $\lambda < \mu$. Then

$$\begin{aligned} f(\mathcal{J}_\mu) &= \mu \\ &= \max(\lambda, \mu) \\ &= \max(f(\mathcal{J}_\lambda), f(\mathcal{J}_\mu)) \\ &= f(\mathcal{J}_\lambda \cap \mathcal{J}_\mu). \end{aligned}$$

Hence, by the minimality of \mathcal{J}_μ ,

$$\mathcal{J}_\mu \subseteq \mathcal{J}_\lambda \cap \mathcal{J}_\mu \subseteq \mathcal{J}_\lambda$$

and therefore $E_\lambda \leq E_\mu$. \square

Lemma 2.5 *f is monotonely continuous, i.e. if $(\mathcal{J}_j)_{j \in J}$ is an increasing net in $\mathcal{D}(\mathcal{R})$ then*

$$f\left(\bigcup_{j \in J} \mathcal{J}_j\right) = \lim_j f(\mathcal{J}_j).$$

Proof: Obviously $\mathcal{J} := \bigcup_{j \in J} \mathcal{J}_j \in \mathcal{D}(\mathcal{R})$. As f is decreasing, $f(\mathcal{J}) \leq f(\mathcal{J}_j)$ for all $j \in J$ and $(f(\mathcal{J}_j))_{j \in J}$ is a decreasing net of real numbers. Hence

$$f(\mathcal{J}) \leq \lim_j f(\mathcal{J}_j).$$

Let $\varepsilon > 0$. Because f is upper semicontinuous, there is $P \in \mathcal{J}$ such that $f(\mathcal{I}) < f(\mathcal{J}) + \varepsilon$ for all $\mathcal{I} \in \mathcal{D}_P(\mathcal{R})$. Now $P \in \mathcal{J}_k$ for some $k \in J$ and therefore

$$\lim_j f(\mathcal{J}_j) \leq f(\mathcal{J}_k) < f(\mathcal{J}) + \varepsilon,$$

which shows that also $\lim_j f(\mathcal{J}_j) \leq f(\mathcal{J})$ holds. \square

Corollary 2.6 *The image of an abstract observable function is compact.*

Proof: Because $\{I\} \subseteq \mathcal{J}$ for all $\mathcal{J} \in \mathcal{D}(\mathcal{R})$ we have $f \leq f(\{I\})$ on $\mathcal{D}(\mathcal{R})$. If $\lambda, \mu \in \text{im} f$ and $\lambda < \mu$ then $\mathcal{J}_\mu \subseteq \mathcal{J}_\lambda$, hence $\bigcup_{\lambda \in \text{im} f} \mathcal{J}_\lambda$ is a dual ideal and therefore contained in a maximal dual ideal $\mathfrak{B} \in \mathcal{D}(\mathcal{R})$. This shows $f(\mathfrak{B}) \leq f$ on $\mathcal{D}(\mathcal{R})$ and consequently $\text{im} f$ is bounded. Let $\lambda \in \overline{\text{im} f}$. Then there is an increasing sequence $(\mu_n)_{n \in \mathbb{N}}$ in $\text{im} f$ converging to λ or there is a decreasing sequence $(\mu_n)_{n \in \mathbb{N}}$ in $\text{im} f$ converging to λ . In the first case we have $\mathcal{J}_{\mu_{n+1}} \subseteq \mathcal{J}_{\mu_n}$ for all $n \in \mathbb{N}$ and therefore for $\mathcal{J} := \bigcap_n \mathcal{J}_{\mu_n} \in \mathcal{D}(\mathcal{R})$

$$f(\mathcal{J}) = \sup_n f(\mathcal{J}_{\mu_n}) = \sup_n \mu_n = \lambda.$$

In the second case we have $\mathcal{J}_{\mu_n} \subseteq \mathcal{J}_{\mu_{n+1}}$ for all $n \in \mathbb{N}$ and therefore $\mathcal{J} := \bigcup_n \mathcal{J}_{\mu_n} \in \mathcal{D}(\mathcal{R})$. Hence

$$f(\mathcal{J}) = \lim_n f(\mathcal{J}_{\mu_n}) = \lim_n \mu_n = \lambda.$$

Therefore $\lambda \in \text{im} f$ in both cases, i.e. $\text{im} f$ is also closed. \square

Step 2 We will now extend $(E_\lambda)_{\lambda \in \text{im} f}$ to a spectral family $E^f := (E_\lambda)_{\lambda \in \mathbb{R}}$. In defining E^f we have of course in mind that the spectrum of the selfadjoint operator A corresponding to E^f should coincide with $\text{im} f$. This forces us to define E_λ for $\lambda \notin \text{im} f$ in the following way. For $\lambda \notin \text{im} f$ let

$$S_\lambda := \{\mu \in \text{im} f \mid \mu < \lambda\}.$$

Then we define

$$E_\lambda := \begin{cases} 0 & \text{if } S_\lambda = \emptyset \\ E_{\sup S_\lambda} & \text{otherwise.} \end{cases}$$

Note that $f(\{I\}) = \max \text{im} f$ and that $\mathcal{J}_{f(\{I\})} = \{I\}$.

Lemma 2.6 E^f is a spectral family.

Proof: The only remaining point to prove is that E^f is continuous from the right, i.e. that $E_\lambda = \bigwedge_{\mu > \lambda} E_\mu$ for all $\lambda \in \mathbb{R}$. This is obvious if $\lambda \notin \text{im}f$ or if there is some $\delta > 0$ such that $] \lambda, \lambda + \delta[\cap \text{im}f = \emptyset$. Therefore we are left with the case that there is a strictly decreasing sequence $(\mu_n)_{n \in \mathbb{N}}$ in $\text{im}f$ converging to λ . For all $n \in \mathbb{N}$ we have $f(\mathcal{J}_{\mu_n}) > f(\mathcal{J}_\lambda)$ and therefore $\mathcal{J}_{\mu_n} \subseteq \mathcal{J}_\lambda$. Hence $\bigcup_n \mathcal{J}_{\mu_n} \subseteq \mathcal{J}_\lambda$ and

$$f\left(\bigcup_n \mathcal{J}_{\mu_n}\right) = \lim_n f(\mathcal{J}_{\mu_n}) = \lambda$$

implies $\bigcup_n \mathcal{J}_{\mu_n} = \mathcal{J}_\lambda$ by the minimality of \mathcal{J}_λ . If $P \in \mathcal{J}_\lambda$ then $P \in \mathcal{J}_{\mu_n}$ for some n and therefore $E_{\mu_n} \leq P$. This shows $\bigwedge_{\mu > \lambda} E_\mu \leq P$. As $P \in \mathcal{J}_\lambda$ is arbitrary we can conclude that $\bigwedge_{\mu > \lambda} E_\mu \leq E_\lambda$. The reverse inequality is obvious. \square

Step 3 Let $A \in \mathcal{R}$ be the selfadjoint operator corresponding to the spectral family E^f . It is obvious from the definition of E^f that

$$\text{sp}(A) \subseteq \text{im}f.$$

The next result shows that the spectrum of A is equal to the image of f .

Lemma 2.7 Let E_λ be constant on the nonempty interval $] \lambda_0, \lambda_1[$. Then

$$\text{im}f \cap] \lambda_0, \lambda_1[= \emptyset.$$

Proof: Because of the right-continuity of E^f we can assume that λ_0 belongs to $\text{im}f$. We show first that $\text{im}f \cap] \lambda_0, \lambda_1[$ consists of at most one element. Assume that $\lambda, \mu \in \text{im}f$, $\lambda_0 < \lambda < \mu < \lambda_1$. Because f is upper semicontinuous we can find, given $\varepsilon \in]0, \lambda - \lambda_0[$, a projection $P \in \mathcal{J}_{\lambda_0}$ such that

$$\forall \mathcal{J} \in \mathcal{D}_P(\mathcal{R}) : f(\mathcal{J}) < \lambda_0 + \varepsilon < \lambda,$$

in particular

$$\lambda_0 = f(\mathcal{J}_{\lambda_0}) \leq f(H_P) < \lambda,$$

hence $\mathcal{J}_\lambda \subseteq H_P \subseteq \mathcal{J}_{\lambda_0}$. This implies

$$P = \inf H_P \leq \inf \mathcal{J}_\lambda = E_\lambda = E_{\lambda_0},$$

hence $E_{\lambda_0} = P \in \mathcal{J}_{\lambda_0}$ and therefore

$$\mathcal{J}_{\lambda_0} = H_{E_{\lambda_0}}.$$

By the same argument, applied to λ and μ , we see that

$$\mathcal{J}_\lambda = H_{E_\lambda} = H_{E_{\lambda_0}} = \mathcal{J}_{\lambda_0}$$

and therefore

$$\lambda_0 = f(\mathcal{J}_{\lambda_0}) = f(\mathcal{J}_\lambda) = \lambda,$$

a contradiction.

We now show that $\lambda_0 \in \text{im} f$ and $]\lambda_0, \lambda[\cap \text{im} f = \emptyset$ imply $\mathcal{J}_{\lambda_0} = H_{E_{\lambda_0}}$.

Choose $\varepsilon > 0$ sufficiently small and choose $P \in \mathcal{J}_{\lambda_0}$ such that

$$\forall \mathcal{J} \in \mathcal{D}_P(\mathcal{R}) : f(\mathcal{J}) < f(\mathcal{J}_{\lambda_0}) + \varepsilon < \lambda.$$

Then $]\lambda_0, \lambda[\cap \text{im} f = \emptyset$ implies

$$\forall \mathcal{J} \in \mathcal{D}_P(\mathcal{R}) : f(\mathcal{J}) \leq \lambda_0,$$

in particular

$$\lambda_0 = f(\mathcal{J}_{\lambda_0}) \leq f(H_P) \leq \lambda_0.$$

This shows $\mathcal{J}_{\lambda_0} \subseteq H_P$ and therefore

$$P = \inf H_P \leq \inf \mathcal{J}_{\lambda_0} = E_{\lambda_0}.$$

Because of $P \in \mathcal{J}_{\lambda_0}$ we therefore have $E_{\lambda_0} \in \mathcal{J}_{\lambda_0}$, i.e. $\mathcal{J}_{\lambda_0} = H_{E_{\lambda_0}}$.

Finally assume that $]\lambda_0, \lambda_1[\cap \text{im} f \neq \emptyset$ and let λ be the unique element of this intersection. Then by the foregoing we obtain $\mathcal{J}_{\lambda_0} = H_{E_{\lambda_0}} = H_{E_\lambda} = \mathcal{J}_\lambda$, i.e. $\lambda_0 = \lambda$, a contradiction. \square

Corollary 2.7 *Let $f : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ be an abstract observable function and let $A \in \mathcal{R}$ be the selfadjoint operator corresponding to the spectral family E^f defined by f . Then $\text{sp}(A) = \text{im} f$.*

Proof: Note that $\lambda \notin \text{sp}(A)$ if and only if E^f is constant on some neighborhood of λ . The definition of E^f shows that $\mathbb{R} \setminus \text{im} f \subseteq \mathbb{R} \setminus \text{sp}(A)$, i.e. $\text{sp}(A) \subseteq \text{im} f$, and the foregoing lemma shows that $\mathbb{R} \setminus \text{sp}(A) \subseteq \mathbb{R} \setminus \text{im} f$, i.e. $\text{im} f \subseteq \text{sp}(A)$. \square

Lemma 2.8 *Let $f : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ be an abstract observable function, A the selfadjoint operator defined by f and $f_A : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ the observable function corresponding to A . Then $f_A = f$.*

Proof: Recall that $f_A(\mathcal{J}) = \inf\{\lambda \mid E_\lambda \in \mathcal{J}\}$ for all $\mathcal{J} \in \mathcal{D}(\mathcal{R})$. Due to $sp(A) = im f_A = im f$ this can be written as $f_A(\mathcal{J}) = \inf\{\lambda \in sp(A) \mid E_\lambda \in \mathcal{J}\}$. If $E_\lambda \in \mathcal{J}$ (with $\lambda \in sp(A)$) then $\mathcal{J}_\lambda \subseteq H_{E_\lambda} \subseteq \mathcal{J}$ and therefore $f(\mathcal{J}) \leq f(\mathcal{J}_\lambda) = \lambda$. This implies

$$f \leq f_A.$$

For the proof of the reverse inequality we distinguish two cases.

(i) Let $\mathcal{J} \in \mathcal{D}(\mathcal{R})$ and let $\lambda = f(\mathcal{J}_\lambda) = f(\mathcal{J})$ be isolated from the right, i.e. $im f \cap]\lambda, \mu[= \emptyset$ for some $\mu > \lambda$. Then, by the proof of the foregoing lemma, $E_\lambda \in \mathcal{J}_\lambda \subseteq \mathcal{J}$ and therefore

$$f_A(\mathcal{J}) \leq \lambda = f(\mathcal{J}).$$

(ii) Let $\lambda = f(\mathcal{J}_\lambda)$ be *not* isolated from the right and let $(\mu_n)_{n \in \mathbb{N}}$ be a strictly decreasing sequence in $im f$ that converges to λ . If $E_\lambda = \inf \mathcal{J}_\lambda \in \mathcal{J}_\lambda$ then $f(\mathcal{J}_\lambda) = f_A(\mathcal{J}_\lambda)$. Let $E_\lambda \notin \mathcal{J}_\lambda$. Let $n \in \mathbb{N}$. Because f is upper semicontinuous there is $P \in \mathcal{J}_{\mu_{n+1}}$ such that

$$\forall \mathcal{J} \in \mathcal{D}_P(\mathcal{R}) : f(\mathcal{J}) < \mu_n.$$

In particular

$$\mu_{n+1} \leq f(H_P) < \mu_n = f(\mathcal{J}_{\mu_n}).$$

Now $f(H_P \cap \mathcal{J}_{\mu_n}) = \max(f(H_P), f(\mathcal{J}_{\mu_n})) = f(\mathcal{J}_{\mu_n})$ and therefore $\mathcal{J}_{\mu_n} \subseteq H_P$ by the minimality of \mathcal{J}_{μ_n} . So we obtain

$$E_{\mu_{n+1}} \leq P \leq E_{\mu_n},$$

hence

$$E_{\mu_n} \in \mathcal{J}_{\mu_{n+1}} \subseteq \mathcal{J}_\lambda.$$

This shows $E_{\lambda+\varepsilon} \in \mathcal{J}_\lambda$ for all $\varepsilon > 0$ and therefore $f_A(\mathcal{J}_\lambda) \leq \lambda$. This proves $f_A = f$. \square

The uniqueness of A is obvious by remark 2.11: if $A, B \in \mathcal{R}_{sa}$ such that $f_A = f = f_B$ then $A = B$.

This completes the proof of theorem 2.6.

The theorem confirms that there is no difference between “abstract” and “concrete” observable functions and therefore we will speak generally of observable functions.

Let $\mathcal{P}_0(\mathcal{R})$ denote the set of nonzero projections in \mathcal{R} . We will now show that observable functions can be characterized as functions $\mathcal{P}_0(\mathcal{R}) \rightarrow \mathbb{R}$ that satisfy a “continuous join condition”. Note that for an arbitrary family $(P_k)_{k \in \mathbb{K}}$ in $\mathcal{P}_0(\mathcal{R})$ we have

$$\bigcap_{k \in \mathbb{K}} H_{P_k} = H_{\bigvee_{k \in \mathbb{K}} P_k}.$$

If $f : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ is an observable function then the intersection property implies

$$f(H_{\bigvee_{k \in \mathbb{K}} P_k}) = \sup_{k \in \mathbb{K}} f(H_{P_k}).$$

This leads to the following

Definition 2.7 *A bounded function $r : \mathcal{P}_0(\mathcal{R}) \rightarrow \mathbb{R}$ is called completely increasing if*

$$r\left(\bigvee_{k \in \mathbb{K}} P_k\right) = \sup_{k \in \mathbb{K}} r(P_k) \quad (2.6)$$

for every family $(P_k)_{k \in \mathbb{K}}$ in $\mathcal{P}_0(\mathcal{R})$.

Note that it is sufficient to assume in the foregoing definition that r is bounded from below because $r(I)$ is an upper bound, in fact the maximum, for an arbitrary increasing function $r : \mathcal{P}_0(\mathcal{R}) \rightarrow \mathbb{R}$.

Because of the natural bijection $P \mapsto H_P$ between $\mathcal{P}_0(\mathcal{R})$ and the set $\mathcal{D}_{pr}(\mathcal{R})$ of principle dual ideals of $\mathcal{P}(\mathcal{R})$ each observable function $f : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ induces by restriction a completely increasing function r_f :

$$\forall P \in \mathcal{P}_0(\mathcal{R}) : r_f(P) := f(H_P).$$

Conversely we will now show that each completely increasing function on $\mathcal{P}_0(\mathcal{R})$ induces an observable function so that we get a one to one correspondence between observable functions and completely increasing functions. This will enable us to complete the proof of theorem 2.5.

Definition 2.8 *Let $r : \mathcal{P}_0(\mathcal{R}) \rightarrow \mathbb{R}$ be a completely increasing function. Then we define a function $f_r : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ by*

$$\forall \mathcal{J} \in \mathcal{D}(\mathcal{R}) : f_r(\mathcal{J}) := \inf_{P \in \mathcal{J}} r(P).$$

Remark 2.12 *It is this definition where we need that $r : \mathcal{P}_0(\mathcal{R}) \rightarrow \mathbb{R}$ is bounded from below. The following example shows that there are functions $r : \mathcal{P}_0(\mathcal{R}) \rightarrow \mathbb{R}$ that satisfy the condition 2.6, but are not bounded from below.*

Let $E = (E_\lambda)_{\lambda \in \mathbb{R}}$ be a spectral family that is bounded from above but not from below. Then $E_\lambda \neq 0$ for all $\lambda \in \mathbb{R}$. Let $M := \min\{\lambda \mid E_\lambda = I\}$. If $P \in \mathcal{P}_0(\mathcal{R})$, then $\{\lambda \leq M \mid P \leq E_\lambda\}$ is a bounded set, and we can define a function $r : \mathcal{P}_0(\mathcal{R}) \rightarrow \mathbb{R}$ by

$$r(P) := \inf\{\lambda \mid E_\lambda \geq P\}.$$

It is easy to see that the proof of the intersection property for observable functions also works in this case, so that we get

$$r\left(\bigvee_{k \in \mathbb{K}} P_k\right) = \sup_{k \in \mathbb{K}} r(P_k)$$

for every family $(P_k)_{k \in \mathbb{K}}$ in $\mathcal{P}_0(\mathcal{R})$. But $r(E_\lambda) \leq \lambda$ for all $\lambda \in \mathbb{R}$, so r is not bounded from below.

It is obvious that

$$\forall P \in \mathcal{P}_0(\mathcal{R}) : f_r(H_P) = r(P)$$

holds.

Proposition 2.13 *The function $f_r : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ induced by the completely increasing function $r : \mathcal{P}_0(\mathcal{R}) \rightarrow \mathbb{R}$ is an observable function.*

Proof: In view of proposition 2.11 we have to show that f_r satisfies

$$f_r\left(\bigcap_{k \in \mathbb{K}} \mathcal{J}_k\right) = \sup_{k \in \mathbb{K}} f_r(\mathcal{J}_k)$$

for all families $(\mathcal{J}_k)_{k \in \mathbb{K}}$ in $\mathcal{D}(\mathcal{R})$. Since f_r is decreasing we have

$$f_r\left(\bigcap_{k \in \mathbb{K}} \mathcal{J}_k\right) \geq \sup_{k \in \mathbb{K}} f_r(\mathcal{J}_k).$$

Let $\varepsilon > 0$ and choose $P_k \in \mathcal{J}_k$ ($k \in \mathbb{K}$) such that $r(P_k) < f_r(\mathcal{J}_k) + \varepsilon$. Now $\bigcap_k H_{P_k} \subseteq \bigcap_k \mathcal{J}_k$, f_r is decreasing and r is completely increasing, hence

$$f_r\left(\bigcap_k \mathcal{J}_k\right) \leq f_r\left(\bigcap_k H_{P_k}\right) = r\left(\bigvee_k P_k\right) = \sup_k r(P_k) \leq \sup_k f_r(\mathcal{J}_k) + \varepsilon$$

and therefore

$$f_r\left(\bigcap_{k \in \mathbb{K}} \mathcal{J}_k\right) \leq \sup_{k \in \mathbb{K}} f_r(\mathcal{J}_k). \quad \square$$

We have formulated theorem 2.6 and the characterization of observable functions by completely increasing functions in the category of von Neumann algebras. A simple inspection of the proofs shows that we have used the fact that the projection lattice $\mathcal{P}(\mathcal{R})$ of a von Neumann algebra \mathcal{R} is a complete orthomodular lattice. Therefore we can translate theorem 2.6 to the category of complete orthomodular lattices in the following way:

Theorem 2.7 *Let \mathbb{L} be a complete orthomodular lattice and let $f : \mathcal{D}(\mathbb{L}) \rightarrow \mathbb{R}$ be an abstract observable function. Then there is a unique spectral family E in \mathbb{L} such that $f = f_E$.*

The function $f_E : \mathcal{D}(\mathbb{L}) \rightarrow \mathbb{R}$ is defined quite naturally as

$$\forall \mathcal{J} \in \mathcal{D}(\mathbb{L}) : f_E(\mathcal{J}) := \inf\{\lambda \in \mathbb{R} \mid E_\lambda \in \mathcal{J}\}.$$

We will present now a further characterization of observable functions. For a function $f : \mathcal{P}_0(\mathcal{R}) \rightarrow \mathbb{R}$ let

$$\mathcal{F}_\lambda := f^{-1}([-\infty, \lambda]) \cup \{0\}.$$

Note that f is lower semicontinuous with respect to the topology of strong convergence if and only if \mathcal{F}_λ is strongly closed in $\mathcal{P}(\mathcal{R})$ for all $\lambda \in \mathbb{R}$.

Proposition 2.14 *Let $f : \mathcal{P}_0(\mathcal{R}) \rightarrow \mathbb{R}$ be a function. Then the following properties are equivalent:*

- (i) *f is completely increasing.*
- (ii) *f is strongly lower semicontinuous and $f(P \vee Q) = \max(f(P), f(Q))$ for all $P, Q \in \mathcal{P}_0(\mathcal{R})$.*
- (iii) *For all $\lambda \in \mathbb{R}$ the set \mathcal{F}_λ is a strongly closed ideal in $\mathcal{P}(\mathcal{R})$.*

Proof: Let f be completely increasing and let $(P_a)_{a \in \mathbb{A}}$ be a net in $f^{-1}([-\infty, \lambda])$ that converges strongly to $P \in \mathcal{P}_0(\mathcal{R})$. Because of $P_a \leq \bigvee_{b \in \mathbb{A}} P_b$ for all $a \in \mathbb{A}$ we have also $P \leq \bigvee_{b \in \mathbb{A}} P_b$ and therefore $f(P) \leq f(\bigvee_{b \in \mathbb{A}} P_b) = \sup\{f(P_a) \mid a \in \mathbb{A}\} \leq \lambda$.

Now assume that (ii) holds. We have to show that \mathcal{F}_λ is an ideal in $\mathcal{P}(\mathcal{R})$ for all $\lambda \in \mathbb{R}$. If $Q \in \mathcal{F}_\lambda$ and $P \leq Q$ then $P \in \mathcal{F}_\lambda$ because f is increasing. If P and Q are two nonzero elements of \mathcal{F}_λ . Then $P \vee Q \in \mathcal{F}_\lambda$ because of $f(P \vee Q) = \max(f(P), f(Q))$.

Finally we show that (iii) implies (i). Let $P, Q \in \mathcal{P}_0(\mathcal{R})$ and $P \leq Q$. From $Q \in \mathcal{F}_{f(Q)}$ we conclude $P \in \mathcal{F}_{f(Q)}$, i.e. $f(P) \leq f(Q)$. Hence f is increasing and therefore $f(P \vee Q) \geq \max(f(P), f(Q))$ for all $P, Q \in \mathcal{P}_0(\mathcal{R})$. Now $P, Q \in \mathcal{F}_{\max(f(P), f(Q))}$ and therefore $P \vee Q \in \mathcal{F}_{\max(f(P), f(Q))}$ because $\mathcal{F}_{\max(f(P), f(Q))}$ is an ideal. This shows $f(P \vee Q) \leq \max(f(P), f(Q))$. Now let $(P_a)_{a \in \mathbb{A}}$ be an arbitrary family in $\mathcal{P}_0(\mathcal{R})$ and let $Q := \bigvee_{b \in \mathbb{A}} P_b$. Then Q is the strong limit of the increasing net $(Q_F)_{F \in \text{Fin}(\mathbb{A})}$ where $Q_F := \bigvee_{a \in F} P_a$ and $\text{Fin}(\mathbb{A})$ denotes the set of all nonempty finite subsets of \mathbb{A} . From $P_a \in \mathcal{F}_{\sup_{b \in \mathbb{A}} f(P_b)}$ for all $a \in \mathbb{A}$, we obtain $Q_F \in \mathcal{F}_{\sup_{b \in \mathbb{A}} f(P_b)}$ for all $F \in \text{Fin}(\mathbb{A})$ and therefore, as $\mathcal{F}_{\sup_{b \in \mathbb{A}} f(P_b)}$ is strongly closed, $Q \in \mathcal{F}_{\sup_{b \in \mathbb{A}} f(P_b)}$. Hence $f(\bigvee_{a \in \mathbb{A}} P_a) = \sup_{a \in \mathbb{A}} f(P_a)$. \square

If \mathcal{R} is a von Neumann algebra, $r : \mathcal{P}_0(\mathcal{R}) \rightarrow \mathbb{R}$ a completely increasing function, $\lambda \in \text{im } r$ and

$$E_\lambda := \bigvee \{P \in \mathcal{P}_0(\mathcal{R}) \mid r(P) = \lambda\}$$

then also $r(E_\lambda) = \lambda$. Hence E_λ is the largest element in the inverse image $r^{-1}(\lambda)$. It is easy to see that

- (i) $\lambda, \mu \in \text{im } r$ and $\lambda \leq \mu$ imply $E_\lambda \leq E_\mu$,
- (ii) and if $(\mu_n)_{n \in \mathbb{N}}$ is a decreasing sequence in $\text{im } r$ converging to $\lambda \in \text{im } r$ then $E_\lambda = \bigwedge_{n \in \mathbb{N}} E_{\mu_n}$

hold. Let $f : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ the observable function induced by r , $E^A = (E_\lambda^A)_{\lambda \in \mathbb{R}}$ the corresponding spectral family and $A \in \mathcal{R}_{sa}$ the selfadjoint operator defined by E^A .

Remark 2.13 *The range of r is dense in $\text{sp}(A)$: $\text{sp}(A) = \overline{r(\mathcal{P}_0(\mathcal{R}))}$.*

Proof: This follows from $f(\mathcal{D}(\mathcal{R})) = \text{sp}(A)$ and $f(\mathcal{J}) = \inf_{P \in \mathcal{J}} r(P)$ for all $\mathcal{J} \in \mathcal{D}(\mathcal{R})$. \square

For $\lambda \notin \text{sp}(A)$ we define E_λ in the very same way as in step 2 of the proof of theorem 2.6. Then $\mathfrak{P}_r := (E_\lambda)_{\lambda \in r(\mathcal{P}_0(\mathcal{R})) \cup (\mathbb{R} \setminus \text{sp}(A))}$ becomes a prespectral family. From corollary 2.4 we know that

$$\forall \lambda \in r(\mathcal{P}_0(\mathcal{R})) : E_\lambda = E_\lambda^A.$$

Hence the foregoing remark and property (ii) show that the spectralization E of \mathfrak{P}_r coincides with the spectral family E^A . So we have proved

Proposition 2.15 *Let $r : \mathcal{P}_0(\mathcal{R}) \rightarrow \mathbb{R}$ be a completely increasing function and let $A \in \mathcal{R}_{sa}$ be the selfadjoint operator determined by r . Then the spectral family E^A of A is the unique extension of the family $(E_\lambda)_{\lambda \in r(\mathcal{P}_0(\mathcal{R}))}$, defined by*

$$E_\lambda := \bigvee \{P \in \mathcal{P}_0(\mathcal{R}) \mid r(P) = \lambda\}.$$

The special case $\mathcal{R} = \mathcal{L}(\mathcal{H})$ deserves a detailed study. Let $\mathcal{P}(\mathcal{L}(\mathcal{H}))$ the lattice of projections in $\mathcal{L}(\mathcal{H})$, $\mathcal{P}_0(\mathcal{L}(\mathcal{H}))$ the subset of nonzero projections and $\mathcal{P}_1(\mathcal{L}(\mathcal{H}))$ the subset of projections of rank one. The decisive feature of the special case $\mathcal{L}(\mathcal{H})$ is that every element of $\mathcal{P}_0(\mathcal{L}(\mathcal{H}))$ is the join of a suitable family in $\mathcal{P}_1(\mathcal{L}(\mathcal{H}))$.

If $r : \mathcal{P}_0(\mathcal{L}(\mathcal{H})) \rightarrow \mathbb{R}$ is a completely increasing function then clearly r is uniquely determined by its restriction to $\mathcal{P}_1(\mathcal{L}(\mathcal{H}))$. Of course not every function $s : \mathcal{P}_1(\mathcal{L}(\mathcal{H})) \rightarrow \mathbb{R}$ is the restriction of a completely increasing function on $\mathcal{P}_0(\mathcal{L}(\mathcal{H}))$: because the representation of $P \in \mathcal{P}_0(\mathcal{L}(\mathcal{H}))$ as the join of a family in $\mathcal{P}_1(\mathcal{L}(\mathcal{H}))$ is far from being unique in general, s must satisfy some compatibility condition (and, as it turns out, some continuity condition too).

The compatibility condition is easy to detect: let P_1, P_2 be two different elements of $\mathcal{P}_1(\mathcal{L}(\mathcal{H}))$ and let $Q \in \mathcal{P}_1(\mathcal{L}(\mathcal{H}))$ be a subprojection of $P_1 \vee P_2$. Then necessarily

$$r(Q) \leq r(P_1 \vee P_2) = \max(r(P_1), r(P_2)).$$

Hence the restriction $s : \mathcal{P}_1(\mathcal{L}(\mathcal{H})) \rightarrow \mathbb{R}$ of a completely increasing function $r : \mathcal{P}_0(\mathcal{L}(\mathcal{H})) \rightarrow \mathbb{R}$ must satisfy

$$\forall P, Q, R \in \mathcal{P}_1(\mathcal{L}(\mathcal{H})) : (P \leq Q \vee R \implies s(P) \leq \max(s(Q), s(R))).$$

Lemma 2.9 *Let $r : \mathcal{P}_0(\mathcal{L}(\mathcal{H})) \rightarrow \mathbb{R}$ be a completely increasing function. Then the restriction s of r to $\mathcal{P}_1(\mathcal{L}(\mathcal{H}))$ is lower semicontinuous with respect to the topology of strong convergence on $\mathcal{P}_1(\mathcal{L}(\mathcal{H}))$.*

Proof: We have to show that for every $\lambda_0 \in \mathbb{R}$

$$s^{-1}([-\infty, \lambda_0]) = \{P \in \mathcal{P}_1(\mathcal{L}(\mathcal{H})) \mid s(P) \leq \lambda_0\}$$

is closed with respect to strong convergence. Let f be the observable function induced by r and let $(E_\lambda)_{\lambda \in \mathbb{R}}$ be the spectral family corresponding to f . If $P \in \mathcal{P}_0(\mathcal{L}(\mathcal{H}))$ then $r(P) = f(H_P) \leq \lambda_0$ if and only if $E_{\lambda_0} \geq P$. Now consider a net $(P_k)_{k \in \mathbb{K}}$ in $s^{-1}(]-\infty, \lambda_0])$ that converges strongly to $P \in \mathcal{P}_1(\mathcal{L}(\mathcal{H}))$. Then $E_{\lambda_0} P_k = P_k$ for all $k \in \mathbb{K}$ and as $E_{\lambda_0} P_k \rightarrow E_{\lambda_0} P$ strongly we conclude $E_{\lambda_0} P = P$, i.e. $P \in s^{-1}(]-\infty, \lambda_0])$. \square

We say that a function $s : \mathcal{P}_1(\mathcal{L}(\mathcal{H})) \rightarrow \mathbb{R}$ induces a completely increasing function $r : \mathcal{P}_0(\mathcal{L}(\mathcal{H})) \rightarrow \mathbb{R}$ if the restriction of r to $\mathcal{P}_1(\mathcal{L}(\mathcal{H}))$ is s .

Theorem 2.8 *A bounded function $s : \mathcal{P}_1(\mathcal{L}(\mathcal{H})) \rightarrow \mathbb{R}$ induces a completely increasing function $r : \mathcal{P}_0(\mathcal{L}(\mathcal{H})) \rightarrow \mathbb{R}$ if and only if the following two conditions are satisfied:*

- (i) *s is lower semicontinuous with respect to the topology of strong convergence on $\mathcal{P}_1(\mathcal{L}(\mathcal{H}))$,*
- (ii) *$s(P) \leq \max(s(Q), s(R))$ for all $P, Q, R \in \mathcal{P}_1(\mathcal{L}(\mathcal{H}))$ such that $P \leq Q \vee R$.*

Proof: A completely increasing function is bounded and therefore its restriction to $\mathcal{P}_1(\mathcal{L}(\mathcal{H}))$ must be bounded too. We have already seen that the conditions (i) and (ii) are necessary.

Conversely, assume that they are fulfilled for a bounded function $s : \mathcal{P}_1(\mathcal{L}(\mathcal{H})) \rightarrow \mathbb{R}$. Let $Q \in \mathcal{P}_0(\mathcal{L}(\mathcal{H}))$ and let $(P_k)_{k \in \mathbb{K}}$ be a family in $\mathcal{P}_1(\mathcal{L}(\mathcal{H}))$ such that $Q = \bigvee_{k \in \mathbb{K}} P_k$. Then we are forced to define

$$r(Q) := \sup_{k \in \mathbb{K}} s(P_k).$$

In order to show that this is well-defined we begin with the case that \mathbb{K} is a finite set.

Let \mathbb{K} be a finite non-empty set and let $Q = \bigvee_{k \in \mathbb{K}} P_k$ with $P_k \in \mathcal{P}_1(\mathcal{L}(\mathcal{H}))$ for all $k \in \mathbb{K}$. Then

$$\sup\{s(P) \mid P \in \mathcal{P}_1(\mathcal{L}(\mathcal{H})), P \leq Q\} = \max_{k \in \mathbb{K}} s(P_k).$$

For the proof we use induction with respect to the size n of \mathbb{K} . For $n = 1$ there is nothing to prove. Let $n > 1$ and assume that the claim is true for all subsets of \mathbb{K} of size $n - 1$. Let $P_k = P_{\mathbb{C}x_k}$ ($k \in \mathbb{K}$), $P = P_{\mathbb{C}x}$ with $x, x_k \in S^1(\mathcal{H})$. Obviously $x = \sum_{k=1}^{n-1} a_k x_k + a_n x_n$ with suitable $a_1, \dots, a_n \in \mathbb{C}$ and therefore $P \leq P_{\mathbb{C} \sum_{k=1}^{n-1} a_k x_k} \vee P_n$. By the induction hypothesis we have

$$s(P) \leq \max(s(P_{\mathbb{C} \sum_{k=1}^{n-1} a_k x_k}), s(P_n)) \leq \max_{k \leq n} s(P_k).$$

Hence $\sup_{P \leq Q} s(P) \leq \max_{k \leq n} s(P_k)$ and the opposite inequality is trivial.

Now let \mathbb{K} be an arbitrary non-empty set, $P = P_{\mathbb{C}x} \leq Q$ and let $Fin(\mathbb{K})$ be the set of all finite non-empty subsets of \mathbb{K} . Then $Q = \bigvee_{F \in Fin(\mathbb{K})} Q_F$, where $Q_F := \bigvee_{j \in F} P_j$, x is the limit of a net of unit vectors $x_F \in \text{lin}_{\mathbb{C}}\{x_j \mid j \in F\}$ and therefore the net $(P_{\mathbb{C}x_F})$ is strongly convergent to P . From the finite case we obtain

$$s(P_{\mathbb{C}x_F}) \leq \max_{j \in F} s(P_j) \leq \lambda_0 := \sup_{k \in \mathbb{K}} s(P_k).$$

Hence $P_{\mathbb{C}x_F} \in \bar{s}^{-1}(]-\infty, \lambda_0])$ for all $F \in Fin(\mathbb{K})$ and therefore $s(P) \leq \lambda_0$ by the lower semicontinuity of s . This shows $\sup\{s(P) \mid P \in \mathcal{P}_1(\mathcal{L}(\mathcal{H})), P \leq Q\} = \sup_{k \in \mathbb{K}} s(P_k)$. Thus r is well-defined and obviously completely increasing. \square

Now we will finish the proof of theorem 2.5. Let \mathcal{R} be an abelian von Neumann algebra and let $f : \mathcal{Q}(\mathcal{R}) \rightarrow \mathbb{R}$ be a continuous function. (f is necessarily bounded because \mathcal{R} is abelian and therefore, due to Stone's theorem, $\mathcal{Q}(\mathcal{R})$ is compact.) Then, using corollary I.3.1 and theorem I.3.1, we have for an arbitrary family $(P_k)_{k \in \mathbb{K}}$ in $\mathcal{P}_0(\mathcal{R})$

$$\begin{aligned} \sup\{f(\mathfrak{B}) \mid \mathfrak{B} \in \mathcal{Q}_{\bigvee_{k \in \mathbb{K}} P_k}(\mathcal{R})\} &= \sup\{f(\mathfrak{B}) \mid \mathfrak{B} \in \overline{\bigcup_{k \in \mathbb{K}} \mathcal{Q}_{P_k}(\mathcal{R})}\} \\ &= \sup\{f(\mathfrak{B}) \mid \mathfrak{B} \in \bigcup_{k \in \mathbb{K}} \mathcal{Q}_{P_k}(\mathcal{R})\} \\ &= \sup_k \sup\{f(\mathfrak{B}) \mid \mathfrak{B} \in \mathcal{Q}_{P_k}(\mathcal{R})\}. \end{aligned}$$

Because of

$$H_P = \bigcap \mathcal{Q}_P(\mathcal{R})$$

for all $P \in \mathcal{P}_0(\mathcal{R})$ it is natural to define

$$r(P) := \sup\{f(\mathfrak{B}) \mid \mathfrak{B} \in \mathcal{Q}_P(\mathcal{R})\}.$$

Then we obtain from the foregoing computation

$$r\left(\bigvee_{k \in \mathbb{K}} P_k\right) = \sup_{k \in \mathbb{K}} r(P_k),$$

i.e. $r : \mathcal{P}(\mathcal{R}) \rightarrow \mathbb{R}$ is a completely increasing function. Let $f_r : \mathcal{D}(\mathcal{R}) \rightarrow \mathbb{R}$ be the corresponding observable function. The following lemma completes the proof of theorem 2.5.

Lemma 2.10 f coincides with the restriction of f_r to $\mathcal{Q}(\mathcal{R})$.

Proof: We have to show that

$$\forall \mathfrak{B} \in \mathcal{Q}(\mathcal{R}) : f(\mathfrak{B}) = \inf_{P \in \mathfrak{B}} r(P)$$

holds.

From the definition of r we see that $f(\mathfrak{B}) \leq m := \inf_{P \in \mathfrak{B}} r(P)$. Let $\varepsilon > 0$. Because f is continuous there is $P_0 \in \mathfrak{B}$ such that $f(\mathfrak{C}) < f(\mathfrak{B}) + \varepsilon$ on $\mathcal{Q}_{P_0}(\mathcal{R})$. Hence

$$m \leq r(P_0) = \sup f(\mathcal{Q}_{P_0}(\mathcal{R})) \leq f(\mathfrak{B}) + \varepsilon.$$

This shows $m \leq f(\mathfrak{B})$. \square

In the next subsection we will show that for abelian von Neumann algebras the bijection $A \mapsto f_A$ from \mathcal{R} onto $C_b(\mathcal{Q}(\mathcal{R}), \mathbb{R})$ is precisely the *Gelfand transformation*. The map $\mathcal{R}_{sa} \rightarrow \mathcal{O}(\mathcal{R})$, $A \mapsto f_A$ is therefore for a general von Neumann algebra a noncommutative generalization of the Gelfand transformation.

2.4 The Gelfand Transformation

Let \mathcal{A} be an abelian von Neumann algebra. Subsequently we will prove that the mapping $A \mapsto f_A$ from \mathcal{A} onto $C(\mathcal{Q}(\mathcal{A}), \mathbb{R})$ is up to the isomorphism $C(\mathcal{Q}(\mathcal{A}), \mathbb{R}) \rightarrow C(\Omega(\mathcal{A}), \mathbb{R})$ the Gelfand transformation of the abelian von Neumann algebra \mathcal{A} .

Let $\sum_{j=1}^m b_j P_j$ be an orthogonal representation of $A \in \text{lin}_\mathbb{C} \mathcal{P}(\mathcal{A})$. By lemma 3.6 in [7],

$$\mathcal{F}_\mathcal{A}(A) := \sum_{j=1}^m b_j \chi_{\mathcal{Q}_{P_j}(\mathcal{A})}$$

is a well defined continuous function on $\mathcal{Q}(\mathcal{A})$. This defines a mapping

$$\mathcal{F}_\mathcal{A} : \text{lin}_\mathbb{C} \mathcal{P}(\mathcal{A}) \rightarrow C(\mathcal{Q}(\mathcal{A})).$$

Proposition 2.16 $\mathcal{F}_\mathcal{A} : \text{lin}_\mathbb{C} \mathcal{P}(\mathcal{A}) \rightarrow C(\mathcal{Q}(\mathcal{A}))$ is an isometric homomorphism of algebras.

Proof: Let $A = \sum_{i=1}^m a_i P_i$ and $B = \sum_{j=1}^n b_j Q_j$ be orthogonal representations of $A, B \in \text{lin}_{\mathbb{C}} \mathcal{P}(\mathcal{A})$. Because $\mathcal{P}(\mathcal{A})$ is distributive we can write

$$\begin{aligned}
 A + B &= \sum_{i=1}^m a_i P_i + \sum_{j=1}^n b_j Q_j \\
 &= \sum_i a_i (P_i Q_1 + \dots + P_i Q_n + P_i (I - (Q_1 + \dots + Q_n))) \\
 &\quad + \sum_j b_j (Q_j P_1 + \dots + Q_j P_m + Q_j (I - (P_1 + \dots + P_m))) \\
 &= \sum_{i,j} (a_i + b_j) P_i Q_j \\
 &\quad + \sum_i a_i P_i (I - (Q_1 + \dots + Q_n)) + \sum_j b_j Q_j (I - (P_1 + \dots + P_m)).
 \end{aligned}$$

This is an orthogonal representation for $A + B$. Applying $\mathcal{F}_{\mathcal{A}}$ gives

$$\begin{aligned}
 \mathcal{F}_{\mathcal{A}}(A + B) &= \sum_{i,j} (a_i + b_j) \chi_{\mathcal{Q}_{P_i Q_j}(\mathcal{A})} \\
 &\quad + \sum_i a_i \chi_{\mathcal{Q}_{P_i (I - (Q_1 + \dots + Q_n))}(\mathcal{A})} + \sum_j b_j \chi_{\mathcal{Q}_{Q_j (I - (P_1 + \dots + P_m))}(\mathcal{A})} \\
 &= \sum_i a_i \chi_{\mathcal{Q}_{P_i}(\mathcal{A}) \cap \mathcal{Q}_{Q_1 + \dots + Q_n}(\mathcal{A})} + \sum_j b_j \chi_{\mathcal{Q}_{Q_j}(\mathcal{A}) \cap \mathcal{Q}_{P_1 + \dots + P_m}(\mathcal{A})} \\
 &\quad + \sum_i a_i \chi_{\mathcal{Q}_{P_i}(\mathcal{A}) \cap \mathcal{Q}_{I - (Q_1 + \dots + Q_n)}(\mathcal{A})} + \sum_j b_j \chi_{\mathcal{Q}_{Q_j}(\mathcal{A}) \cap \mathcal{Q}_{I - (P_1 + \dots + P_m)}(\mathcal{A})} \\
 &= \mathcal{F}_{\mathcal{A}}(A) + \mathcal{F}_{\mathcal{A}}(B).
 \end{aligned}$$

Trivially $\mathcal{F}_{\mathcal{A}}(cA) = c\mathcal{F}_{\mathcal{A}}(A)$ for $A \in \text{lin}_{\mathbb{C}} \mathcal{P}(\mathcal{A})$ and $c \in \mathbb{C}$. A simple calculation shows that $\mathcal{F}_{\mathcal{A}}$ is also multiplicative:

$$\mathcal{F}_{\mathcal{A}}(AB) = \mathcal{F}_{\mathcal{A}}(A)\mathcal{F}_{\mathcal{A}}(B).$$

Let $A = \sum_{i=1}^m a_i P_i$ be an orthogonal representation of $A \in \text{lin}_{\mathbb{C}} \mathcal{P}(\mathcal{A})$. Then

$$|A| = \max_{i \leq m} |a_i|$$

and

$$\left| \sum_i a_i \chi_{\mathcal{Q}_{P_i}(\mathcal{A})} \right|_{\infty} = \max_{i \leq m} |a_i|$$

because the sets $\mathcal{Q}_{P_i}(\mathcal{A})$ are pairwise disjoint. Hence $\mathcal{F}_{\mathcal{A}}$ is isometric. \square

Corollary 2.8 $\mathcal{F}_{\mathcal{A}}$ has a unique extension to an isometric $*$ -isomorphism from \mathcal{A} onto $C(\mathcal{Q}(\mathcal{A}))$. We denote this extension again by $\mathcal{F}_{\mathcal{A}}$.

Proof: It follows from the Stone-Weierstrass-theorem that $\text{lin}_{\mathbb{C}}\{\chi_{\mathcal{Q}_P(\mathcal{A})} | P \in \mathcal{P}(\mathcal{A})\}$ is dense in $C(\mathcal{Q}(\mathcal{A}))$. The unique isometric extension of $\mathcal{F}_{\mathcal{A}}$ to \mathcal{A} is therefore also surjective. \square

Proposition 2.17 $\mathcal{F}_{\mathcal{A}} : \mathcal{A} \rightarrow C(\mathcal{Q}(\mathcal{A}))$ is the Gelfand transformation of the abelian von Neumann algebra \mathcal{A} .

Proof: Let

$$\varepsilon_{\beta} : C(\mathcal{Q}(\mathcal{A})) \rightarrow \mathbb{C}$$

denote the evaluation at the quasipoint $\beta \in \mathcal{Q}(\mathcal{A})$:

$$\forall \varphi \in C(\mathcal{Q}(\mathcal{A})) : \varepsilon_{\beta}(\varphi) = \varphi(\beta).$$

Then for all $P \in \mathcal{P}(\mathcal{A})$

$$\begin{aligned} (\varepsilon_{\beta} \circ \mathcal{F}_{\mathcal{A}})(P) &= \varepsilon_{\beta}(\chi_{\mathcal{Q}_P(\mathcal{A})}) \\ &= \begin{cases} 1 & \text{if } P \in \beta \\ 0 & \text{otherwise} \end{cases} \\ &= \tau_{\beta}(P), \end{aligned}$$

hence $\varepsilon_{\beta} \circ \mathcal{F}_{\mathcal{A}} = \tau_{\beta}$ on a dense part of \mathcal{A} and therefore, by continuity, on all of \mathcal{A} .

The Gelfand transformation

$$\Gamma : \mathcal{A} \rightarrow C(\Omega(\mathcal{A})), \quad A \mapsto \hat{A},$$

is defined by

$$\forall \tau \in \Omega(\mathcal{A}) : \hat{A}(\tau) := \tau(A).$$

The homeomorphism $\theta : \beta \mapsto \tau_{\beta}$ from $\mathcal{Q}(\mathcal{A})$ onto $\Omega(\mathcal{A})$ induces a $*$ -isomorphism

$$\theta^* : C(\Omega(\mathcal{A})) \rightarrow C(\mathcal{Q}(\mathcal{A})), \quad \varphi \mapsto \varphi \circ \theta.$$

We obtain

$$\mathcal{F}_{\mathcal{A}} = \theta^* \circ \Gamma,$$

because

$$\theta^*(\hat{A})(\beta) = \hat{A}(\theta(\beta)) = \hat{A}(\tau_{\beta}) = \tau_{\beta}(A) = \varepsilon_{\beta}(\mathcal{F}_{\mathcal{A}}(A)) = \mathcal{F}_{\mathcal{A}}(A)(\beta)$$

holds for all $A \in \mathcal{A}$ and all $\beta \in \mathcal{Q}(\mathcal{A})$. In this sense $\mathcal{F}_{\mathcal{A}}$ “is” the Gelfand transformation of \mathcal{A} . \square

Theorem 2.9 *Let \mathcal{A} be an abelian von Neumann algebra. Then the mapping $A \mapsto f_A$ from \mathcal{A} onto $C(\mathcal{Q}(\mathcal{A}), \mathbb{R})$ is the restriction of the Gelfand transformation to \mathcal{A}_{sa} .*

Proof: Due to the foregoing proposition we only need to show that $f_A = \mathcal{F}_{\mathcal{A}}(A)$ holds for all $A \in \mathcal{A}_{sa}$. By corollary 2.1 this is true for all $A \in \text{lin}_{\mathbb{R}}\mathcal{P}(\mathcal{A})$. Let A be an arbitrary element of \mathcal{A}_{sa} . We have seen in the proof of theorem 2.4 that f_A is the uniform limit of observable functions f_B with $B \in \text{lin}_{\mathbb{R}}\mathcal{P}(\mathcal{A})$ and by definition $\mathcal{F}_{\mathcal{A}}(A)$ is the uniform limit of functions $\mathcal{F}_{\mathcal{A}}(B)$ with $B \in \text{lin}_{\mathbb{R}}\mathcal{P}(\mathcal{A})$. Hence $f_A = \mathcal{F}_{\mathcal{A}}(A)$. \square

If A is an arbitrary element of the abelian von Neumann algebra \mathcal{A} and $A = A_1 + iA_2$ is its decomposition into selfadjoint parts, then the Gelfand transform of A is

$$\mathcal{F}_{\mathcal{A}}(A) = \mathcal{F}_{\mathcal{A}}(A_1) + i\mathcal{F}_{\mathcal{A}}(A_2).$$

It is therefore natural to define the *complex* observable function of A as

$$f_A := f_{A_1} + if_{A_2}.$$

This definition can be extended to the elements of an arbitrary von Neumann algebra \mathcal{R} .

As an application of our considerations we will characterize *compact normal* operators by its observable functions. We assume that the Hilbert space \mathcal{H} has infinite dimension, for otherwise there is nothing to characterize. Let $A \in \mathcal{L}(\mathcal{H})_{sa}$ be compact. It is well known that A can be represented as

$$A = \sum_{k \in \mathbb{N}} \lambda_k P_{\mathbb{C}e_k}, \quad (2.7)$$

where $\{e_k \mid k \in \mathbb{N}\}$ is a maximal orthonormal set of eigenvectors and the sequence $(\lambda_k)_{k \in \mathbb{N}}$ of eigenvalues converges to zero. The sum converges with respect to the norm.

Now let \mathcal{M} be a maximal abelian von Neumann subalgebra of $\mathcal{L}(\mathcal{H})$ corresponding to a maximal atomic Boolean sector of $\mathcal{P}(\mathcal{L}(\mathcal{H}))$ such that \mathcal{M} contains A and the projections $P_k := P_{\mathbb{C}e_k}$ for all $k \in \mathbb{N}$. Consider the finite-rank approximation

$$A_n := \sum_{k=1}^n \lambda_k P_k$$

of A . The observable function of A_n is

$$f_{A_n} = \sum_{k=1}^n \lambda_k \chi_{\mathcal{Q}_{P_k}}(A) = \sum_{k=1}^n \lambda_k \chi_{\{\beta_{P_k}\}},$$

where $\beta_{P_k} \in \mathcal{Q}(\mathcal{M})$ denotes the atomic quasipoint defined by P_k . This means that f_{A_n} has finite support, contained in $\{\beta_{P_1}, \dots, \beta_{P_n}\}$. In particular, f_{A_n} vanishes on the closed set $\mathcal{Q}(\mathcal{M})_c$ of *continuous* (i.e. non-atomic) quasipoints of $\mathcal{P}(\mathcal{M})$. Since the functions f_{A_n} are the Gelfand transforms of the operators A_n and since the sequence $(A_n)_{n \in \mathbb{N}}$ converges in norm to A , the sequence $(f_{A_n})_{n \in \mathbb{N}}$ converges *uniformly* to the observable function f_A of A . Hence f_A vanishes on $\mathcal{Q}(\mathcal{M})_c$ and, considered as a function on the open set $\mathcal{Q}(\mathcal{M})_{at}$ of atomic quasipoints of $\mathcal{P}(\mathcal{M})$, is an element of $C_0(\mathcal{Q}(\mathcal{M})_{at})$, the algebra of continuous functions $\mathcal{Q}(\mathcal{M})_{at} \rightarrow \mathbb{C}$ that vanish at infinity. Note that $\mathcal{Q}(\mathcal{M})_{at}$ is an open discrete and dense subspace of $\mathcal{Q}(\mathcal{M})$. Therefore $\mathcal{Q}(\mathcal{M})_c$ is the boundary of $\mathcal{Q}(\mathcal{M})_{at}$.

Conversely, let $f : \mathcal{Q}(\mathcal{M}) \rightarrow \mathbb{R}$ be a continuous function that satisfies

- (i) $f|_{\mathcal{Q}(\mathcal{M})_c} = 0$ and
- (ii) $f|_{\mathcal{Q}(\mathcal{M})_{at}} \in C_0(\mathcal{Q}(\mathcal{M})_{at})$

Then f is the uniform limit of a sequence $(f_n)_{n \in \mathbb{N}}$ of functions $f_n : \mathcal{Q}(\mathcal{M}) \rightarrow \mathbb{R}$ of finite support contained in $\mathcal{Q}(\mathcal{M})_{at}$. The selfadjoint operator $A_n \in \mathcal{M}$ is therefore a finite real linear combination of rank-one projections and hence of finite rank. The sequence $(A_n)_{n \in \mathbb{N}}$ converges in norm to the selfadjoint $A \in \mathcal{M}$ that corresponds to f . Hence f is the observable function of the *compact* selfadjoint operator A .

Conditions (i), (ii) are not independent: we show that (i) implies (ii) in a quite general situation.

Lemma 2.11 *Let M be a compact Hausdorff space, $D \subseteq M$ the (discrete open) set of isolated points of M and $X := M \setminus D$. If $f \in C(M)$ vanishes on X , then f vanishes at infinity on D .*

Proof: We assume that D is an infinite set, for otherwise there would be nothing to prove. Let $f : M \rightarrow \mathbb{C}$ be a continuous function that vanishes on X . If $\varepsilon > 0$, we can choose for every $x \in X$ an open neighbourhood U_x of x such that $|f(y)| \leq \varepsilon$ for all $y \in U_x$. The open sets U_x ($x \in X$) together with

the open sets $\{p\}$ ($p \in D$) form an open covering of the compact space M . Hence there are only finitely $p_1, \dots, p_n \in D$ that do not belong to $\bigcup_{x \in X} U_x$. This means $|f(p)| \leq \varepsilon$ for all $p \in D \setminus \{p_1, \dots, p_n\}$, i.e. f vanishes at infinity on D . \square

If the set D of isolated points of M is dense in M , then we can show that condition (ii) implies condition (i):

Lemma 2.12 *Let M be a compact Hausdorff space, $D \subseteq M$ the (discrete open) set of isolated points of M and $X := M \setminus D$. If D is dense in M , then every $f \in C(M)$ that vanishes at infinity on D , vanishes on the boundary X of D .*

Proof: Again we can assume that D is an infinite set. Let $x \in X$ such that $f(x) \neq 0$. We may assume that $f(x) = 1$. Let U_1 be an open neighbourhood of x such that $|f(y)| \geq \frac{1}{2}$ for all $y \in U_1$. Since $\overline{D} = M$, there is some $p_1 \in U_1 \cap D$. Choose a neighbourhood U_2 of x that is contained in U_1 and does not contain p_1 . Then choose $p_2 \in U_2 \cap D$. Proceeding in this way, we generate a sequence $(p_n)_{n \in \mathbb{N}}$ of infinitely many different points in D such that $|f(p_n)| \geq \frac{1}{2}$ for all $n \in \mathbb{N}$. Hence f does not vanish at infinity on D . \square

The set $\mathcal{K}(\mathcal{O}(\mathcal{M}))$ of all continuous functions $f : \mathcal{Q}(\mathcal{M}) \rightarrow \mathbb{C}$ that vanish on $\mathcal{Q}(\mathcal{M})_c$ forms a *selfadjoint* ideal in $C(\mathcal{Q}(\mathcal{M}))$. Also the set $\mathcal{K}(\mathcal{M})$ of all compact operators in \mathcal{M} is a selfadjoint ideal. Summing up, we have proved the following

Proposition 2.18 *Let \mathcal{H} be an infinite dimensional Hilbert space and let \mathcal{M} be a maximal abelian von Neumann subalgebra of $\mathcal{L}(\mathcal{H})$ corresponding to a maximal atomic Boolean sector of $\mathcal{P}(\mathcal{L}(\mathcal{H}))$. Let $\mathcal{Q}(\mathcal{M})_{at}$ be the open discrete (and dense) set of atomic quasipoints of $\mathcal{P}(\mathcal{M})$ and let $\mathcal{Q}(\mathcal{M})_c := \mathcal{Q}(\mathcal{M}) \setminus \mathcal{Q}(\mathcal{M})_{at}$ be the set of continuous quasipoints. Then the restriction of the Gelfand transformation $\mathcal{F}_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{O}(\mathcal{M})$ to the ideal $\mathcal{K}(\mathcal{M})$ of all compact operators in \mathcal{M} is an isometric isomorphism from $\mathcal{K}(\mathcal{M})$ onto the ideal $\mathcal{K}(\mathcal{O}(\mathcal{M}))$ in $\mathcal{O}(\mathcal{M})$ of all $f \in C(\mathcal{Q}(\mathcal{M}))$ that vanish on $\mathcal{Q}(\mathcal{M})_c$ (or, equivalently, vanish at infinity on $\mathcal{Q}(\mathcal{M})_{at}$).*

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